Properties of thermal two-time correlation functions

The goal of this problem is to demonstrate some properties of the (thermal) two-time correlation functions \( h_A(t) h_B(t') \), where
\[
\langle \ldots \rangle \equiv \frac{\text{Tr}(\ldots e^{-\beta H})}{\text{Tr}(e^{-\beta H})}
\]
denotes the thermal average with respect to the Hamiltonian \( H \) \([\beta \equiv (k_B T)^{-1}]\), and \( A(t) \equiv e^{iHt/\hbar} A(0) e^{-iHt/\hbar} \) is an operator in the Heisenberg picture.

(a) Demonstrate the following “time-homogeneity” property of these two-time correlation functions:
\[
\langle A(t)B(t') \rangle = \langle A(t - t')B(0) \rangle = \langle A(0)B(t' - t) \rangle.
\]

(b) Prove the Kubo-Martin-Schwinger identity:
\[
\langle A(t)B(0) \rangle = \langle B(0)A(t + \hbar\beta) \rangle.
\]

(c) Using the results of (a) and (b) show that
\[
(C_S)_{\omega} = \coth \left( \frac{\beta \hbar \omega}{2} \right) (C_A)_{\omega},
\]
where \((C_S)_{\omega}\) and \((C_A)_{\omega}\) are the Fourier transforms of the symmetric \( C_S(t) \equiv \langle \{A(t), A(0)\} \rangle \) and antisymmetric \( C_A(t) \equiv \langle [A(t), A(0)] \rangle \) autocorrelation functions of \( A \), respectively.

Quantum diffusion formalism and optical conductivity

The quantum diffusion formalism offers a theoretical framework for description of charge-carrier transport, even under conditions in which the Boltzmann-equation approach is inapplicable. The central quantity in this formalism is the quantum-mechanical spread
\[
\Delta X^2(t) \equiv \langle [X(t) - X(0)]^2 \rangle,
\]
where \( X(t) \equiv e^{iHt/\hbar} X(0) e^{-iHt/\hbar} \) is the Heisenberg representation of the total position operator and \( \langle \ldots \rangle \) stands for the thermal average with respect to the Hamiltonian \( H \).
(a) Show that
\[ \frac{d}{dt} \Delta X^2(t) = \int_0^t C_S^V(t') dt' , \]
where \( C_S^V(t) \equiv \langle X(t)X(0) + X(0)X(t) \rangle \) is the symmetric autocorrelation function of the velocity operator \( V_X(t) \equiv dX(t)/dt \).

(b) The dissipative part of the optical (dynamic) conductivity at \( \omega \neq 0 \) is given by
\[ \sigma(\omega) = \frac{e^2}{\hbar \nu \omega} \text{Re} \int_0^\infty e^{i\omega t} \langle [V_X(t), V_X(0)] \rangle dt , \]
where \( \nu \) is the system volume. Using the properties of Fourier transforms, derive the following relation between the quantum spread and the optical conductivity:
\[ \sigma(\omega) = -\frac{e^2 \omega^2}{\nu} \frac{\tanh(\beta \hbar \omega/2)}{\hbar \omega} \text{Re} \int_0^\infty e^{i\omega t} \Delta X^2(t) dt . \]