Noninteracting Green’s function for topological insulators (6 Points)

Recall the derivation of the noninteracting Green’s function for a system of fermions. Consider now two-dimensional surface electrons (in the $x-y$ plane) described by the Hamiltonian

$$H_{TI} = \hbar v_F \sum_{k, \alpha, \beta} c_{\alpha k}^\dagger k \cdot \sigma_{\alpha \beta} c_{\beta k},$$

where the indices $\alpha$ and $\beta$ represent the electron spin (↑ or ↓), $\sigma \equiv (\sigma_x, \sigma_y)$ is a vector of Pauli spin matrices, $k \equiv (k_x, k_y)$ is the electron wave vector, and $v_F$ the Fermi velocity. In this problem, we are interested in calculating the Green’s function in the ground state ($T = 0$).

(a) Using the equations of motion for operators in the Heisenberg picture, determine the time-dependence of $c_{\alpha k}(t)$ and $c_{\alpha k}(t')$ ($\sigma \in \{\uparrow, \downarrow\}$) for the noninteracting system described by $H_{TI}$.

(b) Using part (a), show that the noninteracting retarded Green’s function

$$G_{\alpha \beta}^R(k, t - t') \equiv -i \Theta(t - t') \langle [c_{\alpha k}^\dagger(t), c_{\beta k}(t')] \rangle,$$

which in this case is a 2×2 matrix in spin space, at $T = 0$ is given by

$$G^R(k, t - t') = -i \Theta(t - t') e^{-i v_F k \sigma(t-t')}.$$ (3)

(c) Show that in the frequency domain

$$G^R(k, \omega) \equiv \int dt(t - t') e^{i(\omega + i\eta)(t-t')} G^R(k, t - t')$$

$$= \frac{v_F k \cdot \sigma - \omega - i \eta}{v_F^2 k^2 - \omega^2},$$ (4)

where $\eta \to 0^+$.  

Momentum dependence of electron-phonon coupling (4 Points)

Given below are three different types of short-range coupling of a single electron with Einstein phonons (or, more generally, dispersionless bosons) of frequency $\omega$, on a one-dimensional lattice (lattice constant $a = 1$) with $N$ sites. The coupling constants $g$, $\phi$, and $\phi_b$ are dimensionless. For each of these couplings, find the equivalent momentum-space
representation, i.e., transform the corresponding electron-phonon coupling Hamiltonian in real space to the form

$$H_{e-ph} = \frac{1}{\sqrt{N}} \sum_{k,q} \gamma(k, q)a_{k+q}^\dagger a_k(b_{-q}^\dagger + b_q).$$

(5)

Here the $a$’s and $b$’s are the electron- and phonon operators, respectively, while $k$ and $q$ are the corresponding quasimomenta.

(a) Holstein-type (purely local) coupling:

$$H_{e-ph} = g\omega \sum_{i=1}^{N} a_i^\dagger a_i (b_i^\dagger + b_i)$$

(6)

(b) Su-Schrieffer-Heeger (Peierls-type) coupling:

$$H_{e-ph} = \phi\omega \sum_{i=1}^{N} (a_{i+1}^\dagger a_i + \text{h.c.})(b_{i+1}^\dagger + b_{i+1} - b_i^\dagger - b_i)$$

(7)

(c) “breathing” coupling (relevant in cuprate high-$T_c$ superconductors):

$$H_{e-ph} = \phi\omega \sum_{i=1}^{N} a_i^\dagger a_i (b_{i-1/2}^\dagger + b_{i-1/2} - b_{i+1/2}^\dagger - b_{i+1/2})$$

(8)

Here, $i \pm 1/2$ refers to the fact that the Einstein oscillators are placed in the middle between two sites.

Then comment on the differences between these three types of electron-phonon interaction as far as the momentum dependence of the vertex function $\gamma(k, q)$ is concerned.