Theoretical Solid-State Physics, Herbstsemester 2012

Blatt 1

Abgabe: 27. September, 12:00H (Treppenhaus 4. Stock)
Tutor: Patrick Hofer Zi.: 4.13

Die 6 Kreditpunkte für Vorlesung und Übung erhält, wer 50% der Punkte aus den Hausaufgaben erreicht.

(1) Some useful operator identities (5 Punkte)

(a) Show that the commutation relation

\[ [a_m^\dagger a_n, a_k^\dagger a_l] = \delta_{nk} a_m^\dagger a_l - \delta_{ml} a_k^\dagger a_n \]

holds for both bosonic and fermionic operators \( a^\dagger, a \), with \( m, n, k, l \) being arbitrary single-particle quantum numbers, and \( \delta_{nk} \) is the Kronecker delta.

(b) Show that for an arbitrary analytic operator function \( f(A) \) it holds that

\[ e^S f(A) e^{-S} = f(e^S A e^{-S}) , \]

a result that is easily generalized to the case where instead of \( A \) we have several operators.

**Hint:** As a first step, consider the case where \( f(A) = A^n \) with integer number \( n \).

(c) As an application of the general result in part b), show that for the Glauber displacement operator \( D(\beta) \equiv \exp(\beta a^\dagger - \beta^* a) \) \((\beta \in \mathcal{C})\), generating coherent states when acting on the vacuum \( |0\rangle \) of the bosonic mode \( a \) [reminder: \( a|\beta\rangle = \beta|\beta\rangle, |\beta\rangle \equiv D(\beta)|0\rangle \)], it holds that

\[ D^\dagger(\beta)f(a, a^\dagger)D(\beta) = f(a + \beta, a^\dagger + \beta^*) , \]

where \( f(a, a^\dagger) \) is again an arbitrary analytic function.

(2) Bogoliubov transformation (5 Punkte)

The effective Hamiltonian

\[ H = \epsilon_c c^\dagger c + \epsilon_d d^\dagger d - \Delta dc - \Delta^* c^\dagger d^\dagger , \]

contains fermions in two kinds of states \( c \) und \( d \) (i.e., \( \{c, c^\dagger\} = 1, \{c, d\} = 0 \) etc., here, \( \{,\} \) is the anticommutator). We would like to diagonalize this Hamiltonian, i.e., express it in the form

\[ H = E_\gamma \gamma^\dagger \gamma + E_\delta \delta^\dagger \delta + E_0 \]
by introducing the so-called quasiparticle operators $\alpha, \beta$ through the following unitary transformation:

$$c^\dagger = u^* \gamma^\dagger + v \delta , \quad d = -v^* \gamma^\dagger + u \delta .$$

$(u, v$ are complex numbers, $\gamma, \delta$ are fermionic operators, i.e., obey $\{\gamma, \gamma^\dagger\} = 1$ etc. !)

(a) Show that the coefficients have to fulfill $|u|^2 + |v|^2 = 1$.

(b) Express $H$ through $\gamma$ and $\delta$, and determine $u$ and $v$ such that $H$ is diagonalized.

Hint: introduce new variables $\phi, \eta$ by setting $u = \cos \eta, \quad v = e^{i\phi} \sin \eta$.

Determine $\phi, \eta$ such that the “unwanted” terms like $\gamma^\dagger \delta^\dagger$ vanish. Find the energy spectrum of the new quasiparticles, that is, find the expressions for $E_\gamma$ and $E_\delta$ in the special case $\epsilon_c = \epsilon_d = \epsilon$, and $u, v, \Delta$ are real.

(c) Discuss the meaning of $E_\gamma$, $E_\delta$, and $E_0$. 