Quantum-control approach to superconducting qubit arrays

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Outline of the talk

- Introduction to quantum control
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- **Local quantum control of Heisenberg spin-1/2 chains**
  
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- Conclusions and Outlook
Quantum control: generalities

- **State-selective control**: How to steer a quantum system from a given initial state to a pre-determined final state?

- **Operator (state-independent) control**: How to realize a desired unitary transformation (target quantum gate)?

\[ H(t) = H_0 + \sum_{j=1}^{p} f_j(t) H_j \]

\( f_j(t) \) – control fields

The system is **completely controllable** if \( H(t) \) can give rise to an arbitrary unitary transformation on its Hilbert space.
Spin-$\frac{1}{2}$ chain with $XXZ$ Heisenberg interaction

nearest-neighbor $XXZ$ Heisenberg coupling:

$$H_0 = J \sum_{i=1}^{N_s-1} \left( S_{i,x} S_{i+1,x} + S_{i,y} S_{i+1,y} + \Delta S_{i,z} S_{i+1,z} \right)$$

(for definiteness: $J, \Delta > 0$)

Zeeman-like local control Hamiltonian:

$$H_c(t) = \underbrace{h_x(t) S_{1x}}_{f_1(t) H_1} + \underbrace{h_y(t) S_{1y}}_{f_2(t) H_2}$$

total Hamiltonian:

$$H(t) = H_0 + H_c(t)$$
Local control in qubit arrays with “always-on” interactions

conventional control

local control
Complete controllability of an XXZ chain

Acting on the $x$- and $y$-components of the first spin in an XXZ Heisenberg spin chain renders the chain completely controllable!

sufficient to show that the dimension of the Lie algebra $\mathcal{L}_{xy}$ generated by $\{-iH_0, -iS_{1x}, -iS_{1y}\}$ is $d^2 - 1$ ($d \equiv 2^{N_S}$)

$$\Rightarrow \mathcal{L}_{xy} \cong su(d) \Rightarrow e^{\mathcal{L}_{xy}} \cong SU(d) \text{ (complete controllability)}$$
Complete controllability of an $XXZ$ chain

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Generalization: local controllability of quantum networks

D. Burgarth, S. Bose, C. Bruder, and V. Giovannetti, PRA 79, 060305(R) (2009)
Control objectives (target gates)

**CNOT** on the last two qubits of the chain:

\[
\text{CNOT}_{N_s} \equiv \mathbf{I} \otimes \ldots \otimes \mathbf{I} \otimes \left( \begin{array}{c}
|0\rangle\langle0| \otimes \mathbf{I} + |1\rangle\langle1| \otimes \mathbf{X}
\end{array} \right)_{N_s - 2} \text{CNOT}
\]

Flip (**NOT**) of the last qubit in the chain:

\[
\text{X}_{N_s} \equiv \mathbf{I} \otimes \ldots \otimes \mathbf{I} \otimes \mathbf{X}_{N_s - 1}
\]

sufficient to use an \(x\) control field!

√SWAP on the last two qubits of the chain:

\[
\text{SWAP}_{N_s} \quad \text{reminder:} \quad \sqrt{\text{SWAP}} \equiv e^{i \frac{\pi}{8}} e^{-i \frac{\pi}{8}} (\mathbf{X} \otimes \mathbf{X} + \mathbf{Y} \otimes \mathbf{Y} + \mathbf{Z} \otimes \mathbf{Z})
\]
Control pulses and fidelity maximization

alternate $x$ and $y$ (or $x$ only !) piecewise-constant controls:

\[ U(t_f) = U_{y,N_t/2} U_{x,N_t/2} \cdots U_{y,1} U_{x,1} \]

\[ U_{j,n} \equiv e^{-iH_{j,n}T} \quad (j = x, y) \]
Control pulses and fidelity maximization

alternate $x$ and $y$ (or $x$ only !) piecewise-constant controls:

$$h_x(t)$$

$\begin{align*}
h_{x,1} \\
h_{x,2} \\
h_{x,3}
\end{align*}$

full time evolution (total time $t_f \equiv N_t T$):

$$U(t_f) = U_{y,N_t/2}U_{x,N_t/2} \ldots U_{y,1}U_{x,1}$$

$$U_{j,n} \equiv e^{-iH_{j,n}T} \quad (j = x, y)$$

gate fidelity:

$$F(t_f) = \frac{1}{d} \left| \text{tr}[U^\dagger(t_f)U_{\text{target}}] \right|$$

$$\begin{bmatrix} 0 \leq F(t_f) \leq 1 \end{bmatrix}$$

maximize $F = F(\{h_{x,n}; h_{y,n}\})$ numerically for $N_s = 3, 4$
Optimal control pulses for the $X_3$ flip gate

Example results:

1. $N_t = 70, \ T = 0.5 \ J^{-1}$
   
   $F = 1 - 10^{-10}$

2. $N_t = 70, \ T = 0.243 \ J^{-1}$
   
   $F = 1 - 2 \times 10^{-6}$
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For fixed $t_f$, more rapid switching can lead to much higher $F$!

for total time $t_f = 30 \; J^{-1}$:

$N_t = 20, 40, 60, 70$

$F = 0.76, 0.96, 0.999, 1 - 10^{-8}$
Δ ≈ 5 yields shortest times!

For $F \geq 1 - 10^{-3}$

$$t_{\text{CNOT}_3} \approx 11.3 \ J^{-1}$$

$$t_{\text{CNOT}_4} \approx 4.5 \ t_{\text{CNOT}_3}$$
Minimal gate times: Optimal values of anisotropy $\Delta$

$\Delta \approx 5$ yields shortest times!

For $F \geq 1 - 10^{-3}$

$$t_{\text{CNOT}_3} \approx 11.3 \ J^{-1}$$

$$t_{\text{CNOT}_4} \approx 4.5 \ t_{\text{CNOT}_3}$$

for $\Delta \gtrsim 11$ shorter $X_3$ gate times for $x$-only control!

$\Rightarrow$ design principle for superconducting qubits:

$$\frac{E_C}{E_J} \leftrightarrow \Delta$$
“Intrinsic robustness”: CNOT$_3$ gate ($\Delta = 1.0$)

\[ t_f \equiv N_t T = 30J^{-1} \]

\[ h_{j,n} = h_{j,n}^0 \rightarrow \text{randomly chosen } h_{j,n} \in (h_{j,n}^0 - \delta, h_{j,n}^0 + \delta) \]
“Intrinsic robustness”: $\sqrt{\text{SWAP}_3}$ gate ($\Delta = 1.2$)
Saturation value of fidelity

Average fidelity for noisy gate implementation (Nielsen, 2002):

\[
\overline{F}(\varepsilon, U) = \sum_{j=1}^{d} \frac{\text{tr} \left[ U U_j^\dagger U^\dagger \varepsilon(U_j) \right]}{d^2(d + 1)} + d^2
\]

perfect gate implementation: \( \varepsilon : \rho \rightarrow \varepsilon(\rho) = U \rho U^\dagger \Rightarrow \overline{F}(\varepsilon, U) = 1 \)

Equivalent form involving generators \( \{T_j\} \) of \( SU(d) \):

\[
\overline{F}(\varepsilon, U) = \frac{1}{d} + \frac{2}{d(d + 1)} \sum_{j=1}^{d^2-1} \text{tr} \left[ U T_j U^\dagger \varepsilon(T_j) \right]
\]

full randomization: \( \varepsilon : \rho \rightarrow \varepsilon(\rho) = 1_d/d \Rightarrow \overline{F}(\varepsilon, U) = 1/d \)
Spectral low-pass filtering of control fields

practical limitation:
control fields cannot have arbitrarily high frequencies!

frequency-filtered control fields:
\[
\tilde{h}_j(t) = \mathcal{F}^{-1}[f(\omega)\mathcal{F}[h_j(t)]]
\]

ideal low-pass filter:
\[
f(\omega) = \theta(\omega + \omega_0) - \theta(\omega - \omega_0)
\]

\(\omega_0\) – cutoff frequency
Toffoli-gate realization with superconducting qubits

state of the art: two-qubit gates with $F > 90\%$ [DiCarlo et al., (2009)]

TOFFOLI ≡ controlled-controlled-NOT
**ToFelli-gate realization with superconducting qubits**

**state of the art:** two-qubit gates with $F > 90\%$ [DiCarlo et al., (2009)]

TOFFOLI $\equiv$ controlled-controlled-NOT

- trapped-ion [$F \approx 71\%$],
  - photonic [$F \approx 81\%$] realizations in 2009!
- conventional
  - 6 CNOTs + 10 single-qubit operations
  - approach not feasible due to long gate times!
- Way out: **use third level**
  - A. Fedorov et al., arXiv:1108.3966: $F = 64.5 \pm 0.5 \%$
  - M. D. Reed et al., arXiv:1109.4948: $F = 78 \pm 1 \%$
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Can quantum control be of help?

$F \xrightarrow{\text{decoherence}} F \ast \exp(-t_g/T_2)$
Three-qubit (transmon) circuit QED setup at ETH

effective $XY$-type model:

$$H_0 = \sum_{i<j} J_{ij} (\sigma_{ix} \sigma_{jx} + \sigma_{iy} \sigma_{jy})$$

$$J_{12} = J_{23} = J \approx 30 \text{ MHz}, \quad J_{13} \approx 5 \text{ MHz}$$

$$H_c(t) = \sum_{i=1}^{3} \left[ \Omega_{x}^{(i)}(t) \sigma_{ix} + \Omega_{y}^{(i)}(t) \sigma_{iy} \right]$$

$$\sqrt{\Omega_{x}^{2} + \Omega_{y}^{2}} \lesssim 130 \text{ MHz}$$
Toffoli gate in circuit QED: results

cutoff frequency:
\[
\omega_0 = 500 \text{ MHz}
\]
\[
\omega_0 \approx 17J!
\]

\[ t_g \approx 140 \text{ ns} \quad F \approx 99\%
\]
\[ t_g = 75 \text{ ns} \quad F \approx 92\%
\]
Conclusions and Outlook

- Minimal gate times in qubit arrays with $XXZ$-type interaction depend strongly on the anisotropy $\Delta$ [CNOT $\xrightarrow{} \Delta \approx 5$]!

- Large gate fidelities possible in the presence of random errors and for frequency-filtered control fields!

- Using quantum control, within only 75 ns a Toffoli gate can be realized with intrinsic fidelity above 92%!