Journal Club by Oleg Chalaev

One-Dimensional Bose Gases with N-Body Attractive Interactions

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N-particle *attractive* contact interaction

The Hamiltonian:

$$\hat{H} = \int dx \hat{\Psi}^{\dagger}(x) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \hat{\Psi}(x) - \frac{c}{N!} \int dx \hat{\Psi}^{\dagger}(x) \cdots \hat{\Psi}^{\dagger}(x) \hat{\Psi}(x) \cdots \hat{\Psi}(x).$$

Exact ground energy for $N = 2$:

$$\frac{E}{\mathcal{N}_T} = -\frac{mc^2 \left(\mathcal{N}_T^2 - 1 \right)}{24\hbar^2},$$

where $\mathcal{N}_T$ is the total number of particles. We assume

$$c\mathcal{N}_T = const$$

– finite ground state energy per particle.

Mean-field approximation for two-particle interaction.

$$\psi_{GS}(x_1, \cdots, x_{\mathcal{N}_T}) \propto \prod_{i=1}^{\mathcal{N}_T} \psi_0(x_i)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_0 - c \mid \psi_0 \mid^2 \psi_0 = \mu\psi_0,$$

$$E_{GP} = \int dx \psi_0^*(x) \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{c}{2} \mid \psi_0(x) \mid^2\right] \psi_0(x),$$

$$\psi_0(x) = \sqrt{\mathcal{N}_T}\frac{\mathcal{N}}{\cosh(kx)}$$

with $k = mc\mathcal{N}_T/2\hbar^2$ and $\mathcal{N} = (1/2)\sqrt{mc\mathcal{N}_T/\hbar^2}$.

$$\frac{E_{GP}}{\mathcal{N}_T} = -\frac{mc^2\mathcal{N}_T^2}{24\hbar^2},$$

– almost exact result.

Generalization to arbitrary N_T

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-c|\psi(x,t)|^\alpha\right)\psi(x,t),\quad N\equiv\frac{\alpha}{2}+1.$$

time-independent GNLSE equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-c|\psi_0(x)|^\alpha\right)\psi_0(x)=\mu\psi_0(x),\quad \int dx|\psi_0(x)|^2=\mathcal{N}_T.$$

We change the normalization to 1:

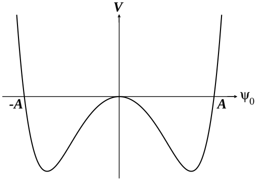
$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}-g|\psi_0(x)|^\alpha\right)\psi_0(x)=\tilde{\mu}\psi_0(x),\quad \int_{-\infty}^{\infty}\psi_0^2(x)dx=1.$$

Connection between stationary and non-stationary solution:

$$\psi_0(x,t)=\psi_0(x-vt)e^{-i(\tilde{\mu}t-vx+v^2t/2)}.$$

Analogy with the Newton equation

$$F = -\tilde{\mu}\psi_0 - g\psi_0^{\alpha+1}, \quad V(\psi_0) = \frac{\tilde{\mu}}{2}\psi_0^2 + \frac{g}{\alpha+2}\psi_0^{\alpha+2}.$$



Solution:

$$\int_A^{\psi_0(x)} \frac{dq}{\sqrt{a^2q^2 - q^{\alpha+2}}} = -\frac{1}{a\alpha}\log\left[\frac{a+\sqrt{a^2-\psi_0^{\alpha}(t)}}{a-\sqrt{a^2-\psi_0^{\alpha}(t)}}\right],$$

$$\psi_0(x) = \frac{A}{\cosh^{\gamma}\left(\frac{\alpha}{2}\sqrt{-2\tilde{\mu}}x\right)}, \quad A = \max|\psi_0(x)|.$$

We also impose the normalization condition.

Normalization condition

$$(-\tilde{\mu})^{\frac{4-\alpha}{2\alpha}} = g^{2/\alpha} \left(\frac{2}{\alpha + 2} \right)^{2/\alpha} \frac{\alpha \Gamma(2/\alpha + 1/2)}{\sqrt{2\pi}\Gamma(2/\alpha)}.$$

When $\alpha \neq 4$, we can use this equation to express $\tilde{\mu}$ as a function of g and, in particular, to write the normalization A as

$$A = \left(\sqrt{\frac{2g}{\pi\gamma(\gamma+1)}} \frac{\Gamma(\gamma + 1/2)}{\Gamma(\gamma)} \right)^{\frac{2}{4-\alpha}} .$$

For $\alpha = 4$ – infinite degeneracy:

$$\forall \tilde{\mu} \quad \psi_0(x) = \left(\frac{\sqrt{-3\tilde{\mu}/g}}{\cosh(2\sqrt{-2\tilde{\mu}}x)} \right)^{1/2}$$

– but only for *the critical interaction strength* $g^* = \frac{3\pi^2}{8}$.

Infinite degeneracy at $g = g^*$

For $g > g^*$ – collapse. For $g < g^*$ – unlocalized solution.

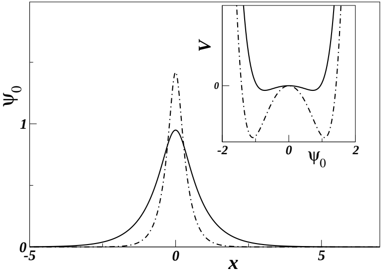


Figure: Wavefunction for $N = 3$ and $g = g^*$ plotted for $\tilde{\mu} = -1$ (solid line) and $\tilde{\mu} = -5$ (dot-dashed line). Inset: corresponding potential for $\tilde{\mu} = -1$ (solid line) and $\tilde{\mu} = -5$ (dot-dashed line).

Stability diagram

From variational approach:

$$D\alpha = 4$$

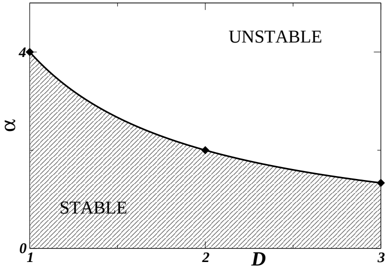


Figure: Stability diagram: $D = 1, 2, 3$ corresponds respectively to $\alpha = 4, 2, 4/3$, i.e. $N = 3, 2, 5/3$.

Conclusions

- The stability of the generalized non-linear Schrödinger equation is studied.
- $\alpha = 4$ (3-particle interactions) is critical in 1D.
- Infinite degeneracy of the ground state at the critical interaction value.

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