Contents

Part I

Basics of the disorder averaging technique

Chapter 1

Introduction

This is the public version of my notes. It contains "undefined references" to some unpublished stuff which I didn't include here.

These notes are available on the web.

"Неофициальность" этих записок значит, что они нуждаются в "зачистке" (которая потихонечку идёт вместе с добавлением нового материала). Если наткнётесь на ошибку или глупость, пожалуйста, сообщите мне.

These are my notes(=Aufzeichnungen, Notizen): calculations, proofs that are not written in my PhD thesis and papers. Even if you print this (rather large) document, I suggest that you have it opened in Acrobat Reader simultaneously. This is because this .pdf file contains many hyperlinks which facilitate surfing through it.

Take a look on the following mesoscopic experiments: 0803.0568. A nice book is [1] – covers almost all topics of the mesoscopics. (Sometimes I am arguing with it, however.) What is written here is close to [2]. See also [3, 4, 5, 6, 7], 0310012, 0402203.

1.1 Introduction

In these notes, we consider metals with isotropic Fermi surface, which in reality is almost never true, as one can see on this webcite. See p. [8]31 about this. Typical values of E_F, p_F, D, τ, L_φ, τ_φ, etc. can be found in §[9]11.4 and on p. [cond-mat/0412664]4.

For a definition of a mesoscopic conductor, see p. [8]200. Roughly speaking, a conductor, which is smaller than the dephasing length, is called a *mesoscopic* conductor[1]:

$$L_T, L_\varphi \gtrsim L, \quad L_T = \sqrt{\frac{\hbar D}{2\pi k T}} \tag{1.1}$$

STOP The restrictions for both L_T and L_φ come because of the interaction. so what is the difference? (L_φ is the coherence length, L_T is the thermal length.[2]) This means that a mesoscopic system is typically small and cold:

$$L \sim 5 \div 40\mu\text{m}, \quad T \sim 0.1\text{K}, \quad \tau_D = L^2/D = 1 \div 60\text{ns}.$$

Our main tool in calculations is the disorder averaging technique [5, 10]. We study systems with randomly placed impurities forming random potential $U(r)$ for electrons. All impurities are identical. A common value for the mean scatttering (on impurities) free path is $l \sim 100$Å. This means that a metal (or a semiconductor) which we study is really "dirty" in comparison with the "clean" (ballistic[3]) case, where $l \sim 10\mu$m, see p. [11]90. In addition, we always suppose that[4]

$$p_F l \gg \hbar \Longleftrightarrow \mu\tau \gg \hbar, \quad p_F \stackrel{\text{df}}{=} \sqrt{2m\mu}, \tag{1.2}$$

[1] For the clear definition of L_φ see, e.g. papers of Montambaux. Do not mix up L_φ and L_{in}; the latter can be much larger, see sec. **??**.

[2] The importance of L_T is discussed on p.[4]42, but this explanation seems me just wave-handing. The effect of L_T, if it exists, would appear automatically, e.g., in my conductivity calculations neglecting the interaction. BTW, L_T is not mentioned among the relevant length scales in [1].

[3] It is interesting how this approach is changed in the limit $\tau \to \infty$, see cond-mat/0611523.

[4] On p. [12]483 (§95), (1.2) is the requirement of the quasiclassical approximation. On the other hand, this is just the condition for the absence of strong (Anderson) localization.

where μ is the (T-dependent) chemical potential. We would also like to assume that $T \ll \hbar/\tau$ – only this will allow us, e.g., to approximate momentum operators as $\hat{\vec{p}} \approx p_F \hat{\vec{n}}$, where $\hat{\vec{n}}$ is the direction operator, $|\vec{n}| = 1$. Also in 3D this allows us to forget about energy-dependence of the DoS. BTW, τ itself is also energy and temperature[5] dependent.

STOP Relations between energy scales, see (1.19):

$$\text{distance between neighbouring levels} = \delta = \hbar v_F/L \ll T \ll \hbar/\tau_0 \ll E_F. \tag{1.3}$$

This model of disorder was first studied by I. M. Lifshitz (И. М. Лифшиц) [14] see p. [15]62; it is different from the Anderson model where potential holes (or bariers) with random depth (height) are placed in the cites of a regular lattice.

The potential $U(\vec{r})$ is supposed to be δ-correlated[6]:

$$\overline{U(\vec{r})U(\vec{r}')} \stackrel{\text{df}}{=} \frac{\hbar^2 \delta(\vec{r}-\vec{r}')}{2\pi\nu_0\tilde{\tau}}, \tag{1.4}$$

where the overbar $\overline{\cdot\cdot\cdot}$ denotes averaging over different disorder configurations, $\tilde{\tau}$ is the parameter characterizing the strength of disorder, and ν_0 is the DoS (defined on p. 21) at the Fermi level[7]. Eq. (1.4) reflects the assumption that the potential of every single impurity is independent from the others. Due to the fact that the number of impurities is very large, there must be a sort of a Central Limit Theorem for the potential function $U(r)$ (which is a sum of contributions from all impurities) saying that it has Gaussian distribution ([17]20). As a consequence, the moment of arbitrary order $\overline{GG\ldots G}$ averaged over such a potential can be expressed in terms of first $\bar{G}$ and second $\overline{GG}$ moments.[8]

- The highest priority have notations from [5], then those from [19]. The symbol ♣ in an equation means that the equation is to be checked.
- d denotes the (quasi)dimension, L_i denotes size of the system in ith direction, $V = \prod_j L_j$ is the volume, μ is the (T-dependent) chemical potential,[9] p_F is the Fermi momentum,[10] $p_F = m v_F$, m is the electron effective mass, v_F is the Fermi velocity, $e < 0$ is the charge of an electron. The standart (i.e. superconducting) flux quantum $\varphi_0 = \pi/|e|$. Let us define

$$\forall \vec{a},\vec{b} \in \mathbb{C}^d \quad \vec{a}\circ\vec{b} \equiv (a_1 b_1, a_2 b_2, \ldots, a_d b_d)^T, \quad \vec{a}\left(\vec{b}\circ\vec{c}\right) = \left(\vec{a}\circ\vec{b}\right)\vec{c},$$
$$\vec{a}/\vec{b} \equiv (a_1/b_1, a_2/b_2, \ldots, a_d/b_d)^T, \quad \overleftrightarrow{ab} = \{a_i b_j\}_{i,j=1}^d \quad \text{(diadic)}.$$

  I use the mathematical notation $\mathrm{Supp}\,[f(x)] \equiv \left\{x \,\middle|\, f(x) \neq 0\right\}$ in the approximate sence: $\mathrm{Supp}\,[f(x)]$ means the set of $\{x\}$ where $f(x)$ differs sufficiently from zero. The following square root ($\sqrt{z}$) and argument branches ($\arg z$) is chosen by default:

$$\sqrt{Re^{i\varphi}} = \sqrt{R}e^{i\varphi/2}, \quad R > 0, \quad -\pi < \varphi \equiv \arg z \leq \pi \quad \Rightarrow \sqrt{i} \equiv (1+i)/\sqrt{2}.$$

- $(x,y)^n \equiv \sum_{m=0}^n a_m x^m y^{n-m}$ with arbitrary/unknown (and independent on x and y) coefficients $\{a_m\}_{m=0}^n$. Used in the relation symbols to indicate precision, e.g., $O[(x_a,x_b)^6] \equiv O(x_a,x_b)^6$ in (7.13). More generally,

$$(x_1,x_2,\ldots,x_m)^n = \sum_{i_1=0}^{n}\sum_{i_2=0}^{n-i_1}\cdots\sum_{i_{m-1}=0}^{n-\sum_{k=1}^{m-2} i_k} a_{i_1,i_2,\ldots,i_{m-1}} x_1^{i_1} x_2^{i_2}\ldots x_{m-1}^{i_{m-1}} x_m^{n-\sum_{r=1}^{m-1} i_r}.$$

- Note on graphical representation: for all 3 types of Green's functions used here, $G \equiv G_C$, G_R and G_A it is assumed $x \longrightarrow x' = G(x,x')$ and $x \longleftarrow x' = G(x',x)$. Correspondingly, $p \longrightarrow p' = G(p,p')$ and $p \longleftarrow p' = G(p',p)$;

[5] I mean phonons which actively contribute to elastic scattering, see p. [13]471.

[6] See Fig.8.3 on p.[16]300; See also ([cond-mat/0305478]3.8-11) for more general case. Also arXiv/0801.1786. See also STOP condmat of F. Guinea in 2008, where it is claimed that in case of long-range (logarithmic) disorder *conductivity can be inverse proportional to the concentration of impurities.*

[7] In the problems taking spin into account, we assume that ν is the same, as in the spinless case. That is, ν is the DoS *per spin.*

[8] This is true only within the ZLA, see Sec. 3.6 and 8.3. Also VK told me that this is incorrect. If it would be so, then tunnelling DoS would have Gaussian distribution which is not correct in general case. The problem is discussed in [18]. However, I don't see what's wrong in my reflection apart from the fact that Central Limit Theorem that I know is not for random functions, but for random variables.

[9] In these notes I (historically) also often use the symbol E_F, which in most occasions should be substituted with μ.

[10] p_F is derived from μ and thus also has temperature dependence.[20] Another source of its temperature dependence is the Landau Fermi liquid theory. STOP Is it true that $v_F = \left.\frac{\partial\varepsilon}{\partial p}\right|_{p=p_F}$ =group velocity?

- Pauli matrices: σ^{α}_{ij}, where $\alpha = 0\ldots3$ and σ^0 is the unity matrix.

  The variables p, k, q are used for momenta, ξ is energy variable (usually denote deviation from μ), ω is the frequency.

  Note: in the past p, k, q had dimension cm^{-1}; I did not yet change this (inserting $\hbar$ where appropriate) completely. The same mess I might have with ξ. All (including those energetic and momentum) constants assumed to have their usual CGSE values. Also all other quantities have their normal CGS dimensions.

  The gauge with scalar potential equal to zero is used. We define the current operator in momentum space without $\propto \vec{A}$ term[11]:

$$\hat{\vec{j}} = e\hat{\vec{v}} = \frac{ie}{\hbar}[\hat{H}, x] = \boxed{\text{without SOI}} = -i\frac{e\hbar}{m}\vec{p} \approx -i\frac{ep_{\mathrm{F}}}{m}\vec{n} \equiv -ie\vec{v} \equiv -iev\vec{n}, \quad v = p_{\mathrm{F}}/m. \tag{1.5}$$

- The equations and figures from the books/papers are citated with the corresponding number from the bibliography: e.g., ([21]2.22) stands for the equation (2.22) from [21].

- Instead of saying, e.g., "diagram in Fig. 3.1(a)" I say "diagram 3.1(a)".

In the following, we introduce averages $\overline{G_{\mathrm{R/A}}}$ and $\overline{G_{\mathrm{R}}G_{\mathrm{A}}}$ over the ensemble of samples. All higher order averages (like e.g. $\overline{G_{\mathrm{R}}G_{\mathrm{R}}G_{\mathrm{A}}G_{\mathrm{A}}}$) are expressed as infinite series of these ones and additional lonely impurity lines, e.g. like in case of Hikami boxes in (13.35) and (13.36).

In some occasions, I use indices "I" and "II" in order to distinguish between one and the same operator in the first and second quantization.

1.2 Notations and abreviations

$\sqrt{i} \equiv (1+i)/\sqrt{2}$, $\xi = p^2/(2m) - E_{\mathrm{F}}$, $\Re, \Im$ = real and imaginary part, $g = p_{\mathrm{F}}l/\hbar = 2E_{\mathrm{F}}\tau/(2\hbar) \gg 1$, see (3.41), AE=asymptotic expansion, BC=boundary conditions, BTW=by the way, CCV=charge current vertex, CD=cooperon or diffuson, CS=coordinate system, DL=Daniel Loss, DA=diffusion approximation, DAT=disorder averaging (diagrammatic) technique, DF=distribution function, DM=density matrix or Dmitry Maslov, DoS=density of states, EG=electron gas, EM=Evgeni Mischenko (Евгений Мищенко), FI=functional integral, FQ=first quantization, GF=Green's function, GFB=Green functions box (I mean any box made of Green function lines: bubbles, triangles, squares, etc), IAL=impurity averaging lines, IMHO=in my humble opinion, KE=kinetic equation, lhs=left hand side, NLSM=non-linear sigma-model, ME=matrix element, MF=magnetic field, MSH=Mischchenko, Shytov, Halperin, see sec.7.3, OP=one-particle, OS=Oleg Shalaev, (Олег Шалаев – myself), QED=quantum electrodynamics, QKE=quantum kinetic equantion, QM=quantum mechanics (or mechanical), rhs=right hand side, RMT=random matrix theory, s.c.=so called, S.e.=Schrödinger equation, SCBA=СБП=self-consistent Born approximation, sec. 1.4, SHC=spin-Hall conductivity, SOI=spin-orbit interaction, SPM=stationary point method (=?=метод стационарной фазы), SQ=second quantization, VG (Виталик, ВГ) =Vitaly Golovach (Виталий Головач), VK (ВК) =Vladimir Kravtsov (Владимир Кравцов), VRD=vertex renormalization diagram, WL=weak localization, WF=wave function, WR=Wigner representation=mixed coordinate-momentum representation, ZBA=zero-bias anomaly (in the tunneling DoS), ZLA=zero loop approximation, ♣=to be rechecked, ☺=thanks to.

Русские сокращения: ВАХ=вольтамперная характеристика, ВФ=волновая функция, КТП=квантовая теория поля, МП=матрица плотности, МЭ=матричный элемент, СФ=собственная функция, ФИ=функциональный интеграл, функциональное интегрирование, ФП=фазовый переход.

1.3 Basics

The Fouirier tranformation on a finite interval:

$$f(x) = \frac{1}{L}\sum_{k_n} e^{ik_n x} f(k_n), \quad f(k_n) = \int_0^L \mathrm{d}x e^{-ik_n x} f(x), \quad \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} = \lim_{L\to\infty} \frac{1}{L}\sum_{k_n}, \tag{1.6}$$

[11] That is, there is no direct coincidence between (1.5) and (3.2). In the presence of the SOI, see (6.7),6.8 and Sec. 7.2. An alternative way of calculating velocity operator is to apply Legendre transformation to the Hamiltonian, so that $v_\alpha = \frac{\partial H}{\partial p_\alpha}$.

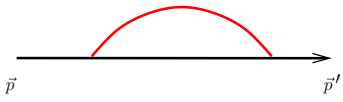
where $k_n = 2\pi n/L$, $n \in \mathbb{Z}$. From (1.6) one gets also the correspondence between δ - functions in discrete and continous k - space: $L\delta_{n,0}$ corresponds to $2\pi\delta(k)$.

$$\forall \vec{p} = 2\pi\vec{n}/\vec{L} \qquad \int \mathrm{d}^d r e^{i\vec{p}\vec{r}} = V\delta_{\vec{p},0}.$$

Physical magnitudes are usually given by $\int \mathrm{d}E f(E)G(\lambda, E)$; sometimes $f(E) \equiv 1$, sometimes it is some distribution function. The shift by chemical potential μ permits to write Green's functions in a (1.9) - like form and also eliminate μ from the equilibrium (i.e. Fermi) and sometimes (since deviations from equilibrium are usualy near the μ - level) nonequilibrium distribution functions. This does not damage the diagram technique.

Note that since in diagram technique in the coordinate space one should use $U(x) = U(\vec{r})\delta(t)$ for time independent potential (see [5], p. 118), the dimension of the potential in the momentum space is length^d/time (i.e. the same as the dimension of U_0).

1. If one-variable function does not depend on time: $f(x) = f(\vec{r})$, then in Fourier space $f(q) = 2\pi\delta(\omega)f(\vec{q})$.
2. If two-variable function depends only on difference of coordinates: $G(x,x') = G(x-x') = \int \frac{\mathrm{d}^{d+1}p}{(2\pi)^{d+1}} e^{ip(x-x')}G(p)$, then $G(p,p') = \int \mathrm{d}^{d+1}x\mathrm{d}^{d+1}x' e^{-ipx+ip'x'}G(x,x') = (2\pi)^{d+1}\delta(p-p')G(p)$.
3. $G^T(\vec{r},\vec{r}\,') \equiv G(\vec{r}\,',\vec{r})$; $G^T(\vec{p},\vec{p}\,') \equiv G(-\vec{p}\,',-\vec{p})$.
4. Two - variable Matsubara (-Green) function in case of conserving-energy interaction reduces to $\mathfrak{G}(E_n,E_l) = \frac{1}{T}\delta_{nl}\mathfrak{G}(E_n)$.
5. If we consider external potential as a perturbation and there is no spatial homogeneity, in coordinate and Fourier space:
$$\delta_1 G(x,x') = \int \mathrm{d}^{d+1}y G(x,y)V(y)G(y,x'), \quad \delta_1 G(p,p') = \int \frac{\mathrm{d}^{d+1}k_1\mathrm{d}^{d+1}k_2}{(2\pi)^{2(d+1)}} G(p,k_1)V(k_1-k_2)G(k_2,p').$$
In case of spatial homogeneity the latter reduces to $\delta_1 G(p,p') = G(p)V(p-p')G(p')$.
6. A simple first order exchange diagram in the interaction case:



$$= \int \frac{\mathrm{d}^{d+1}q\mathrm{d}^{d+1}q_1\mathrm{d}^{d+1}q_2}{(2\pi)^{3(d+1)}} G(p,q_1+q)G(q_1,q_2)G(q_2+q,p')V(q).$$

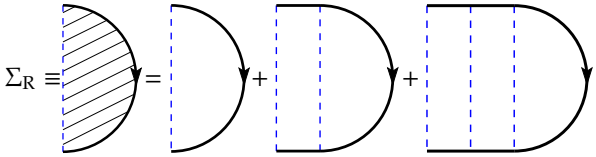
1.4 Average of a Green's function

A single scattering event is described in [22]... In this section we calculate the disorder average of a GF, $\overline{G_{\mathrm{R/A}}}$; later (see Sec. 2) we also calculate $\overline{G_{\mathrm{R}}G_{\mathrm{A}}}$. It is correct[12] what is told in [5], beginning from ([5]39.7) and till $\beta = \mathrm{sign}\,\beta/(2\tau)$.

The disorder strongly affects properties of the system, starting from its GFs. A GF of the disordered system can not be approximated with a first few orders of perturbation expansion; one has to sum the entire (infinite) series.[13]

At this point it is pertinent to explain why the diagrams with crossed impurity lines can be disregarded. The following is not a general proof[14] (which would be valid for an arbitrary diagram), but just a comparison of two diagrams in Fig. 1.1(b).

After averaging one realizes that series for $\bar{G}^{(0)}(\vec{p}) \equiv G(\vec{p})$ implies the following self energy[15] [SCBA , see ([5]39.7) and sec. 3.7]:



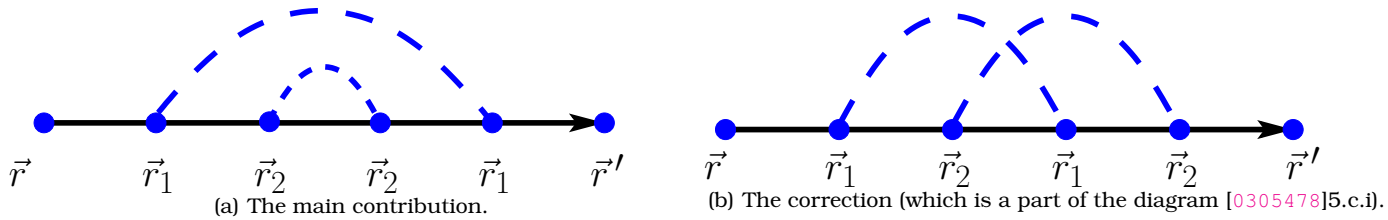
$$\frac{n}{V\hbar^2}\sum_{p_1}|U(\vec{p}-\vec{p}_1)|^2 G_{\mathrm{R}}(\vec{p}_1) + \frac{n^2}{(V\hbar^2)^2}\sum_{p_1,p_2}|U(\vec{p}-\vec{p}_1)|^2|U(\vec{p}-\vec{p}_1-\vec{p}_2)|^2 G_{\mathrm{R}}^2(\vec{p}_1)G_{\mathrm{R}}(\vec{p}_2) + \ldots, \quad \vec{q}_1 = \vec{p}-\vec{p}_1, \quad \vec{q}_2 = \vec{p}-\vec{p}_1-\vec{p}_2. \tag{1.7}$$

[12] Except one thing: to my opinion in ([5]39.6) there should be $-\frac{i\,\mathrm{sign}\,\omega}{2\tau}$ instead of $\frac{i\,\mathrm{sign}\,\omega}{2\tau}$.

[13] Note: the perturbation theory for G_{R} and G_{A} is the same as for casual function G because the disorder potential is time-independent [so there is no frequency (energy) transfer in momentum space].

[14] I have a feeling that it is not hard to make it general, though.

[15] Validity: according to ([0305478]3.10b), SCBA is justified if conductance $\gg e^2/h$; otherwise corrections like diagram 1.1(b) become important.



(a) The main contribution.

(b) The correction (which is a part of the diagram [0305478]5.c.i).

Figure 1.1: Two possible diagrams for $\bar{G}$.

The expression for $\Sigma_{\rm R}$ can be obtained from (1.7) by changing index "R"→"A". The expression for the self energy of the time-ordered GF Σ can be obtained from (1.7) by removing index "R". For arbitary realistic impurity potential $U(\vec{p})$ every term in (1.7) is finite. Note that momenta variables $\vec{q}_i$, $i \geq 1$ are "large": $\forall i\ q_i \sim p_{\rm F}$ in (1.7).

Why do we ignore diagrams with crossed IAE, e.g., 1.1(b), in the ladder-series (1.7)?

Bad (hand-waving) explanation: The values of momentum in diagrams for the average Green function obey two types of limitations. The first type stands for laws of momentum conservation which are strict. The second is the requirement that all Green functions momenta have absolute values in the range[16] $p \in (p_{\rm F} - 1/l, p_{\rm F} + 1/l)$. Let us compare 2 diagrams in Fig. 1.1 for fixed value of external momentum $\vec{p}$. The condition that "internal" modules of momentum must be close to $E_{\rm F}$ in case of the diagram in Fig. 1.1(a) leads to the system of 2 approximate equations:

$$p_1 \approx p_{\rm F}, \quad p_2 \approx p_{\rm F},$$

while the same condition applied to the diagram in Fig. 1.1(b) leads to the system of three approximate equations:

$$p_1 \approx p_{\rm F}, \quad p_2 \approx p_{\rm F}, \quad |\vec{p}_1 + \vec{p}_2 - \vec{p}| \approx p_{\rm F}.$$

From this we conclude that, independent of dimensionality, the right diagram is $(p_{\rm F} l)$ times smaller than the left one[17].

Better explanation: We see that a diagram with "crossings" will always be small if we consider it in the frequency-coordinate space, using the coordinate-space expressions (1.25) for the averaged GF. We see that coordinate arguments are always "entangled" in a diagram with "crossings", and are disentangled in a "ladder-like" diagram. Because of that the integrand in the expression for an "entangled" diagram will contain an oscillating factor $\sim \exp[i\vec{p}\vec{r}/\hbar]\exp[-r/l]$, while the integrand for a "ladder-like" diagram contains only decaying exponent $\sim \exp[-r/l]$. The integration result will be then $pl/\hbar$ times smaller for an "entangled" diagram compared to a "ladder" diagram. It is important to remember that p is determined by the frequency argument ω of the GF: $p = \sqrt{2m(E_{\rm R} + \omega)}$. In case of a non-degenerate 3DEG say, in a semiconductor, for low-energy part of the relevant electrons $pl \lesssim \hbar$ so that the loop expansion is not valid. People don't care about that and just use (IMO, totally unjustified in this case) Boltzmann equation, e.g., to calculate spin relaxation rate and obtain meaningful results. I would be happy to resolve this puzzle...

O.k, that was all about "entangled" diagrams, let's move on. Next we define

$$\Im\Sigma_E^{R/A} \stackrel{\rm df}{=} \mp\frac{1}{2\tau_E}, \tag{1.8}$$

where I want to consider general case, when τ depends on energy.[18] It is clear that such a dependence would result in temperature dependence of physical quantities expressed in terms of GFs. The average of a Green function will have the form[19]:

$$G(p) = \frac{1}{E - \xi_{\vec{p}} + \frac{i}{2\tau_E}\frac{E}{|E|}}, \quad G_{\rm R/A}(p) = \frac{1}{E - \xi_{\vec{p}} \pm \frac{i}{2\tau_E}}, \quad \tau_E \in \mathbb{R}. \tag{1.9}$$

[16] That is, we don't expect that our expression for $\bar{G}_{\rm R/A}$ is valid for $p \ll p_{\rm F} - 1/l$, i.e., deep under the surface of the Fermi sphere. So, we are unable to calculate the *non-universal contributions*, see Sec. 3.7.1.

[17] In the case of diffuson and cooperon the proof is actually the same, as for $\bar{G}$ – we must just "unbend" corresponding diagrams.

[18] We always assume that τ is *momentum-independent*; this is a self consistent and absolutely neccessary assumption.

[19] It is easy to demonstrate, that, generally $G_{\rm R}^E = G_{\rm A}^{E\dagger}$; see p. [23]82. One notes that this holds in the simpliest case (1.9) and in more complicated case (6.17).

In (1.9) $\xi_{\vec{p}} = \varepsilon(\vec{p}) - E_F$ = energy of the particle shifted by the Fermi energy.[20] Note however, that in the left part of the kinetic equation ([21]2.60) stands $\Re\Sigma$. This I STOP don't understand: apparently, according to (1.9) and (1.8) there should stand $i\Im\Sigma$! This concerns also the discussion about $\Re\Sigma$ is ignored in τ, see below. See Sec. 3.7.3.

Let us derive τ_E. The result will be the same for two models:

1. The limit of the hard-sphere model with the sphere radius $\to 0$. The potential will be

$$U(\vec{r}) = U_0\left[\sum_{a=1}^{N}\delta(\vec{r}-\vec{r}_a) - n\right], \quad \tilde{\tau} \stackrel{\text{df}}{=} \left(2\pi\nu_0 n U_0^2/\hbar^2\right)^{-1}, \quad n \equiv \frac{N}{V}, \tag{1.10}$$

$$\int \mathrm{d}r U(\vec{r}) = 0, \text{ or, in momentum space, } U(q=0) = 0,$$

$$\overline{U(\vec{r})} \equiv \frac{1}{V^N}\int \mathrm{d}r_1 \ldots \mathrm{d}r_N U(\vec{r}) = 0, \quad \overline{U(\vec{r})U(\vec{r}')} = nU_0^2[\delta(\vec{r}-\vec{r}') - n].$$

2. Arbitrary white-noise δ-correlated potential. {see generalizations: ([cond-mat/0305478]3.8)} Here $\tilde{\tau}$ is defined by

$$\overline{U(\vec{r})U(\vec{r}')} \stackrel{\text{df}}{=} \frac{\hbar^2}{2\pi\nu_0\,\tilde{\tau}}\delta(\vec{r}-\vec{r}'). \tag{1.11}$$

Historically I learned the first one first:

$$\Sigma_{\mathrm{R/A}}(\vec{q}_{\vec{m}}, E) \equiv \Sigma_E(\vec{q}_{\vec{m}}) = \frac{n}{V}\sum_{\vec{n}}\frac{|U(\vec{q}_{\vec{m}} - \vec{p}_{\vec{n}})|^2}{E - \varepsilon(\vec{p}_{\vec{n}}) - \Sigma_E(\vec{p}_{\vec{n}})} = \frac{nU_0^2}{V}\sum_{\vec{n}}\frac{1}{E - \varepsilon(\vec{p}_{\vec{n}}) - \Sigma_E(\vec{p}_{\vec{n}})}, \tag{1.12}$$

In the approximation with the constant DoS[21] $\nu(\xi) \approx \nu_0$, (1.12) is equivalent to considering only the first term[22] in (1.7):

$$\Im\Sigma_{\mathrm{R}}^{(0)} = nU_0^2\int_{-\infty}^{\infty}\mathrm{d}\xi\nu_0 \times \Im\frac{1}{E - \xi + i\delta}, \quad \delta = +0, \tag{1.14}$$

$$\text{From (1.14) it follows :} \quad \tau_E = \tau_0 = \tilde{\tau}. \tag{1.15}$$

The E-dependent part of $\Im\Sigma_E$ can be estimated as[23] $\frac{E}{\tau E_F} \ll E$. In short, we can think that[24] $\mathrm{Re}\Sigma_E = 0$.

The imaginary part of (1.12) gives us an important sum rule for $\boxed{\Im\Sigma_E \equiv -\frac{1}{2\tau_E}}$:

$$\forall E \quad 2\pi\nu_0\tilde{\tau} = \frac{\hbar^2}{nU_0^2} = \int_{-\infty}^{\infty}\mathrm{d}\xi\frac{\nu(\xi)}{(E-\xi)^2 + \frac{1}{4\tau_E^2}} = \int_{-\infty}^{\infty}\mathrm{d}\xi\frac{\nu(\xi+E)}{\xi^2 + \frac{1}{4\tau_E^2}} = \frac{1}{V}\sum_{\vec{n}\in\mathbb{Z}^d}G_{\mathrm{R}}(\vec{p}_{\vec{n}}, E)G_{\mathrm{A}}(\vec{p}_{\vec{n}}, E), \tag{1.16}$$

[20] It seems to me that VK once has told me that $\varepsilon(\vec{p}) \neq \frac{\hbar^2p^2}{2m}$, however in [10] (and e.g., also in [24]) it is written that $\varepsilon(\vec{p}) = \frac{\hbar^2p^2}{2m}$. Also DL is convinced that $\varepsilon(\vec{p}) = \frac{\hbar^2p^2}{2m}$. If $\Re\Sigma$ essentially depends on $\vec{p}$, then $\varepsilon(\vec{p}) \neq \frac{\hbar^2p^2}{2m}$. Note that if $\varepsilon(\vec{p}) = \frac{\hbar^2p^2}{2m}$, then in 2D nothing saves us from the divergence of $\Re\Sigma$ in (1.12), since $\nu(\xi)$ is approximately a constant. However, a non-parabolicity in semiconductors can occur due to the Kane model, see, e.g., PRB**73**113314.

[21] See pp. [25]28-37. We define ν as spinless DoS, so that it is the same in case of spinful and spinless electrons: (Ω_0 is the complete solid angle; in 3D $\Omega_0 = 4\pi$, in 2D $\Omega_0 = 2\pi$).

$$\nu(\xi) \equiv \frac{1}{V}\sum_{\vec{n}}\delta\left(\xi + \frac{\mu}{\hbar} - \frac{\varepsilon(\vec{p}_{\vec{n}})}{\hbar}\right) \quad \text{so that} \quad \frac{1}{V}\sum_{\vec{p}} = \int\mathrm{d}\xi\nu(\xi)\frac{\mathrm{d}\Omega}{\Omega_0}. \tag{1.13}$$

Another way is to say that $\nu_E = \frac{\partial N_E}{\partial E}$, where N_E is the number of levels inside the sphere $E < E_F$, defined from $N_E(2\pi)^d/V = V_E$, where V_E is the volume of the sphere in momentum space with the sphere radius p_s such that $\varepsilon(p_s) = E$. In 2D $V_E = 2mE\pi$, and in 3D $V_E = (2mE)^{3/2}4\pi/3$. The two definitions differ by the factor V.

[22] It is easy to see that due to the analytical properties the other ones = 0. Due to the same reason we can substitute $G \to G^{(0)}$ in (1.7).

[23] Here E appears because we want it to go to 0 when $E \to 0$, τ - because it appears due to the averaging over impurities, and E_F serves just for dimension. $(\tau E_F)^{-1} \sim 0.02$.

[24] Если бы это было не так, можно бы было попробовать засунуть его в Фурье - преобразование: $f(x) = \int\frac{\mathrm{d}^dp\mathrm{d}E}{(2\pi)^{d+1}}e^{i\vec{p}\vec{r}-i(E-\mu-\mathrm{Re}\Sigma_E)t}f(p)$. Но непонятно: а можно ли вообще так сдвигать преобразование Фурье? Ясно, что есть по меньшей мере 1 частный случай, когда это недопустимо: если $\mathrm{Re}\Sigma_E = E$. Вдобавок положение усугубляется тем, что в отличие от $\Im\Sigma_E \equiv -\frac{1}{2\tau_E}$ вполне возможно, что приближение $\mathrm{Re}\Sigma_E = \mathrm{const}$ недопустимо, и связано это может быть с расходимостью $\int\mathrm{d}\xi$ в уравнении для $\mathrm{Re}\Sigma_E$, аналогичному (1.16).

From (1.16) we see that out of the constant DoS approximation

$$\frac{\tilde\tau}{\tau_0}=\frac{\tilde\nu}{\nu_0},\quad \tilde\nu\stackrel{\rm df}{=}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\pi}\frac{\nu(z/(2\tau_0))}{1+z^2}.$$

Given (1.9), from (3.33) we have

$$\frac{1}{V}\sum_{\vec n\in\mathbb{Z}^d} G_{\rm R}(\vec p_{\vec n},E)G_{\rm A}(\vec p_{\vec n},E)=2\pi\nu_E\tau_E, \tag{1.17}$$

so that from (1.16) one concludes that

$$\forall E\quad \nu_E\tau_E=\nu_0\tau_0;\quad \tau_0=\tilde\tau,\ \text{so that}\ \tilde\nu=\nu_0=\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\pi}\frac{\nu(z/(2\tau_0))}{1+z^2}, \tag{1.18}$$

so that the even part of $\delta\nu_E\stackrel{\rm df}{=}\nu_E-\nu_0$ changes its sign, if not $\delta\nu_E\equiv 0$. This should hold for a non-interacting system so that there is no surprise that this claim is not valid, e.g., for the zero bias anomaly.

Let us parametrize[25] dependencies of ν_E and τ_E in the vicinity of point $E=0$:

$$\nu_E\approx\nu_0(1+xE)=\nu_0+\delta\nu_E,\quad \tau_E\approx\tau_0(1-yE),\quad |E|\lesssim T^*/2,\quad xT^*/2\ll 1,\quad yT^*/2\ll 1. \tag{1.19}$$

We assume that energy-dependent deviation in (1.19) is small for ν: $\delta\nu_E\ll\nu_0$ and for all other quantities: τ_E, D_E (see (13.5)), etc. Since integrals $\int_{-\infty}^{\infty}\mathrm{d}\xi$ converge on $\xi\sim\hbar/\tau$, we also have to impose $x\ll 1$. (Otherwise the loop expansion is not valid, see Sec. 3.6.)

Already from (1.18) it follows that $x=y$, but historically I've deduced it in another way:
Если мы попробуем посчитать поправку к τ жульническим способом [т.е. разложим $\nu(\xi)$ в (1.14) вплоть до линейного члена], то получим:

$$\Im\Sigma=\frac{nU_0^2}{\hbar^2}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\Im G^{(0)}_{\rm R/A}(\vec p)\equiv\mp\frac{i}{2\tau_E},\quad \frac{1}{\tau_E}=2\pi\nu_0 nU_0^2\left(1+E\frac{\nu_0'}{\nu_0}\right), \tag{1.20}$$

Or we can choose a more honest way by expanding (1.16) in powers of E. This leads to

$$\int_{-\infty}^{\infty}\mathrm{d}\xi\nu(\xi)\left[\frac{\nu_0'(\xi)/\nu(\xi)}{\xi^2+\frac{1}{4\tau_0^2}}+\frac{\tau_0'/\tau_0}{2\tau_0^2\left(\xi^2+\frac{1}{4\tau_0^2}\right)^2}\right]=0, \tag{1.21}$$

$$\nu_0'(\xi)/\nu(\xi)\approx x,\quad \tau_0'/\tau_0=-y,\quad \nu(\xi)\approx\nu_0.$$

From both (1.20) and (1.21) it follows that $x=y$ so that $\nu_E\tau_E=\nu_0\tau_0$. (1.22)

Expanding (1.16) up to second order in E one can see that still $\nu_E\tau_E=\nu_0\tau_0$.

Now some notes about the derivation of (13.5). For small q we use an approximation $\varepsilon(\vec p+\vec q)=\varepsilon(\vec p)+\vec v\vec q+bq^2/2$. Before perfoming the integration over ξ one should expand the integrand in powers of E, ω and $\vec q$. Then for the zero-order term (1.16) is applied; in other terms one can use (1.19) for $\nu(\xi)$. To make $\sum_{\vec n}G_{\rm R}(\vec p_{\vec n},E)G_{\rm A}(\vec p_{\vec n}-\vec q_{\vec m},E-\omega)$ invariant under arbitrary shift in $\vec n\in\mathbb{Z}^n$, and in particular, to maintain the obvious relation $\sum_{\vec n}G_{\rm R}(\vec p_{\vec n},E)G_{\rm A}(\vec p_{\vec n}-\vec q_{\vec m},E-\omega)=\sum_{\vec n}G_{\rm R}(\vec p_{\vec n}+\vec q_{\vec m},E)G_{\rm A}(\vec p_{\vec n},E-\omega)$, one has to assume $b=xD_0/\tau_0=v^2x/d$, so that

$$\varepsilon(\vec p+\vec q)=\varepsilon(\vec p)+\vec v\vec q+x\frac{D_0}{2\tau_0}q^2=\varepsilon(\vec p)+\vec v\vec q+x\frac{v^2}{2d}q^2. \tag{1.23}$$

From (1.23) it folllows that usually, when we are not interested in the effects due to the ξ-dependence of the DoS ν, we can drop quadratic term in the decomposition of $\varepsilon(\vec p+\vec q)$.

For future purposes we may need the generalization of DoS (1.13):

$$\nu(\xi,v)\equiv\frac{1}{V}\sum_{\vec n}\delta\left(\xi+\mu-\varepsilon(\vec p_{\vec n})\right)\delta\left(v+v_{\rm F}-\left\|\vec\nabla\varepsilon(\vec p_{\vec n})\right\|\right),\quad \int_{-v_{\rm F}}^{\infty}\mathrm{d}v\nu(\xi,v)=\nu(\xi).$$

[25] An example where one need to introduce see (1.19) - like dependence is thermoelectric effect, see- [8], p. 103.

1.5 More about Born approximation

See also PRB**53**1850.

In this Section, we consider the 3D case, since in 2D $G^\omega(r)$ is a Hankel function, which we don't like. It is known that $(4\pi r)^{-1}$ is the 3D Green function for the Laplace operator δ. [In particular, one can check that $\forall r \neq 0\ \Delta(4\pi r)^{-1} = 0$.] The equation for the GF of a free electron gas is the Helmholz equation ([19]7.7a):

$$\left(k^2 + \delta\right) G^\omega(\vec{r} - \vec{r}') = -\frac{2m}{\hbar^2}\delta(\vec{r} - \vec{r}'), \quad k = \sqrt{2m(\mu + \omega\hbar)}/\hbar, \quad \mu \equiv \mu(T), \tag{1.24}$$

where the minus sign coincides with math-physics definition, but is opposite to the [5]-standard (which we follow). One checks that $\forall r \neq 0\ \Delta e^{ikr}/(4\pi r) = -k^2 e^{ikr}/(4\pi r)$, so that

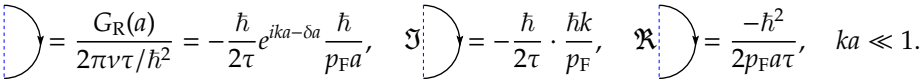
$$G(r) = -\frac{2m}{\hbar^2}\frac{\exp(ikr - \delta r)}{4\pi r}, \tag{1.25}$$

where $\delta = +0$ for the electron gas in the absence of disorder; Eq. (1.25) is valid also for the disorder-averaged GF in the SCBA if one assumes that δ is finite (it can be also ω-dependent).

$$\begin{aligned} G(p) &= \int \mathrm{d}^3 r e^{-i\vec{p}\vec{r}} G(r) = \\ &= -\frac{2m}{\hbar^2}\cdot 2\pi \int_0^\infty \mathrm{d}r r^2 \int_{-1}^1 \mathrm{d}z e^{-iprz}\frac{\exp(ikr - \delta r)}{4\pi r} = -\frac{2m}{\hbar^2}\int_0^\infty \mathrm{d}r r \frac{1}{-2ipr}\left(e^{-ipr} - e^{ipr}\right)\exp(ikr - \delta r) = \\ &= -\frac{2m}{\hbar^2}\frac{1}{-2pi}\left[\frac{1}{ip + ik - \delta} - \frac{1}{-ip + ik - \delta}\right] = \frac{2m}{\hbar^2}\left[k^2 - p^2 + 2ik\delta - \delta^2\right]^{-1}, \quad \frac{\delta\hbar k}{m} \equiv \frac{1}{2\tau} \end{aligned}$$

From (1.24) we conclude that τ inherits the temperature-dependence from the chemical potential μ. The product $\nu\tau$ has to be constant so that (1.11) is temperature-independent.

Following DM, the simplest Born self energy is just $G(a)$, where a =size of an impurity (which saves $\Re\Sigma$ from divergence):



$$= \frac{G_R(a)}{2\pi\nu\tau/\hbar^2} = -\frac{\hbar}{2\tau}e^{ika - \delta a}\frac{\hbar}{p_F a}, \quad \Im = -\frac{\hbar}{2\tau}\cdot\frac{\hbar k}{p_F}, \quad \Re = \frac{-\hbar^2}{2p_F a\tau}, \quad ka \ll 1. \tag{1.26}$$

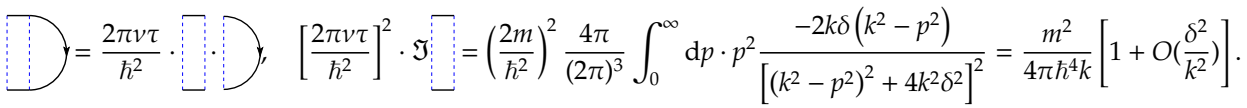
where a is the size of an impurity (a cut-off for large momenta). Eq. (1.26) gives the first-order Born correction to the self energy, if the GF line is the one for the free electron gas (with $\delta = +0$). If δ is finite, (1.26) gives the self-consistency equations. In case when $\omega = 0$, these are

$$G_R(\vec{p}) = \left[G_R^{(0)}(\vec{p}) - \Sigma\right]^{-1}, \quad \Sigma = = \frac{G_R(a)}{2\pi\nu\tau/\hbar^2}, \quad \Longrightarrow k^2 = k_0^2 + \frac{1}{al}, \quad \frac{1}{\tau_\omega} = \frac{\hbar k}{p_F}\cdot\frac{1}{\tau}.$$

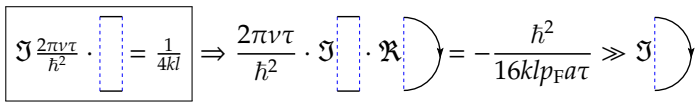
Thus τ in the averaged GF has a weak $\sim \omega/E_F$ frequency dependence. This dependence may be neglected near the Fermi level; there are cases, however, when it is important – e.g., in the calculation of the DM-norm in (**??**), which defines p_F. Thus we *can not* evaluate (**??**), so that even at $\omega = 0$ it is not obvious that $k = p_F$ and $\tau_{\omega=0} = \tau$.

Более того, усреднённые ГФ зависят от ФР посредством импульса Ферми $p_F = \sqrt{2m\mu}$ (в равновесии ФР определяется температурой, и p_F зависит от T). Замечу, что это противоречит духу (хотя и не букве) высказываний между формулами ([26]16,17).

The second-simplest diagram we calculate in the momentum representation:

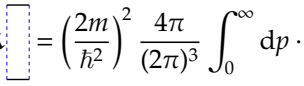


$$= \frac{2\pi\nu\tau}{\hbar^2}\cdot \cdot , \quad \left[\frac{2\pi\nu\tau}{\hbar^2}\right]^2\cdot\Im = \left(\frac{2m}{\hbar^2}\right)^2\frac{4\pi}{(2\pi)^3}\int_0^\infty \mathrm{d}p\cdot p^2\frac{-2k\delta\left(k^2 - p^2\right)}{\left[(k^2 - p^2)^2 + 4k^2\delta^2\right]^2} = \frac{m^2}{4\pi\hbar^4 k}\left[1 + O(\frac{\delta^2}{k^2})\right].$$



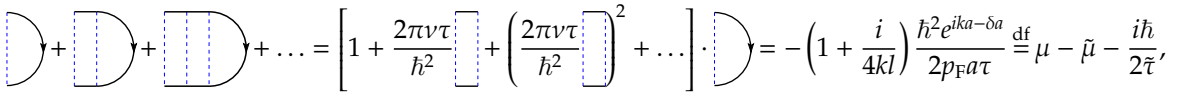
$$\Rightarrow \boxed{\Im\ \frac{2\pi\nu\tau}{\hbar^2}\cdot = \frac{1}{4kl}} \Rightarrow \frac{2\pi\nu\tau}{\hbar^2}\cdot\Im \cdot \Re = -\frac{\hbar^2}{16klp_F a\tau} \gg \Im \quad \text{in case when } 8klp_F a \ll \hbar.$$

Next,



$$\left[\frac{2\pi\nu\tau}{\hbar^2}\right]^2\cdot\Re = \left(\frac{2m}{\hbar^2}\right)^2\frac{4\pi}{(2\pi)^3}\int_0^\infty \mathrm{d}p\cdot p^2\frac{(k^2 - p^2)^2 - 4k^2\delta^2}{\left[(k^2 - p^2)^2 + 4k^2\delta^2\right]^2} = \frac{m^2\delta}{2\pi\hbar^4 k^2}\left[1 + O(\frac{\delta^2}{k^2})\right] \to 0, \quad \frac{\delta}{k} \to 0.$$

For simplicity let us consider the limiting case when $kl \gg 1$ and $8klp_Fa \ll \hbar$ and $\omega = 0$. The Born ladder can be exactly summed:

$\Sigma^R =$  $= -\left(1 + \frac{i}{4kl}\right)\frac{\hbar^2 e^{ika-\delta a}}{2p_F a\tau} \stackrel{\text{df}}{=} \mu - \tilde{\mu} - \frac{i\hbar}{2\tilde{\tau}},$

where $\tilde{\mu}$ is the renormalized chemical potential, and $\tilde{\tau}$ is the renormalized mean scattering time:

$$\tilde{\mu} = \mu + \frac{\hbar^2}{2p_F\tau}\left(\frac{1}{a} - \frac{1}{4l}\right) \gg \mu, \quad \frac{\hbar}{2\tilde{\tau}} = \frac{\hbar^2}{2p_F\tau}\cdot\left[\frac{1}{4kla} + k\right] \Rightarrow \tilde{\tau} \ll \tau \text{ if } k \sim p_F. \tag{1.27}$$

We also want to take into account more sophisticated rainbow diagrams, e.g.,  . In order to do this we have to start from the beginning of this Sec. with renormalized parameters:

$$\hbar k \to \hbar\tilde{k}, \quad \delta \to \frac{1}{2\tilde{k}l}\left[\frac{1}{4kla} + k\right].$$

 $= \frac{G_R(a)}{2\pi\nu\tau/\hbar^2} = -\frac{\hbar}{2\tau}\frac{\hbar\exp\left[ia\tilde{k}\right]}{p_F a}, \quad \tilde{k} = \frac{1}{\sqrt{la}}, \quad |\Im$  $| = \frac{\hbar}{2\tau}\cdot\frac{\hbar}{p_F\sqrt{la}} \gg \frac{\hbar}{2\tau},$

where the GF-line stands for the GF with the renormalized parameters (1.27). We continue the procedure more and more so that k_n grows and we don't know what to do with that:

$$k_{n+1}^2 = k_n^2 - \frac{2m}{\hbar^2}\Re\Sigma_n = k_n^2 + \frac{1}{la} - \frac{1}{l^2} \approx k_n^2 + \frac{1}{la}, \quad k_{n+1}\delta_{n+1} = k_n\delta_n - \frac{m}{\hbar^2}\Im\Sigma_n \Rightarrow \delta_{n+1} = \frac{\delta_n + \frac{1}{2l}\left[\frac{1}{4k_n^2 la} + 1\right]}{\sqrt{1 + \frac{1}{k_n^2 la}}}.$$

We conclude that in the limit of small a

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \delta_n = \infty.$$

In reality, the growth could be stopped when the condition $k_n l \gg 1$ is (no?) longer valid. (The condition $\delta_n/k_n \ll 1$ will not be violated.)

It is faster to use the SCBA: assume that exact (i.e. sum of ALL diagrams without crossings) GF looks like $G(p) = (2m/\hbar^2)(k^2 - p^2 - \Sigma)$, where $\Sigma = \hbar^2/(2\pi\nu\tau)\int d^3p/(2\pi)^3 G(p)$. If we denote $\tilde{k}^2 = k^2 - Re\Sigma$ and $1/(2\tau) = -\Im\Sigma$, then the GF can be rewritten as $G(p) = (2m/\hbar^2)(\tilde{k}^2 - p^2 + im/(\hbar\tau))$.  Stop reading here.

Now let us calculate the corrections to the Born approximation arising from diagrams with crossings. Following DM, the corresponding correction to the free energy Σ_ω^R in the momentum space is

 $= -\left(\frac{m}{2\pi\hbar^2}\right)^3\left(\frac{\hbar^2}{2\pi\nu\tau}\right)^2\int d^3r\frac{\exp(3i\tilde{k}r)}{r^3}e^{-i\vec{p}\vec{r}} = -\frac{\hbar}{2\tau}\cdot\frac{1}{4\pi}\cdot\frac{\hbar}{p_F l}\cdot 2\pi\int_0^\infty dr\frac{\exp(3i\tilde{k}r)}{r}\frac{\sin pr}{pr}$

The expression becomes p-dependent only for $p \gtrsim |\tilde{k}|$; the large-p asymptotic is:

$\Im$  $= -\frac{\hbar}{2\tau}\cdot\frac{\pi}{2}\cdot\frac{3\hbar}{2p_F l}\frac{\Re\tilde{k}}{p}, \quad p \gg |\tilde{k}|.$

Thus this crossed diagram contributes to the p-dependence of τ. This dependence contains smallness $\propto \hbar/(p_F l)$.

Consider the simplest diagram with crossing in real space:

$$\int d^3x e^{-i\vec{p}\vec{x}}\frac{\hbar^2}{(2\pi\nu\tau)^2}\int d^3r_1 d^3r_2 G(r - r_1)G^3(r_1 - r_2)G(r - x - r_2),$$

where we used the symmetry of G: $G(r_1 - r_2) = G(r_2 - r_1)$.

$$\frac{\hbar^2}{(2\pi\nu\tau)^2}\frac{1}{(4\pi)^4}\int d^3x e^{-i\vec{p}\vec{x}}\int d^3r_1 d^3(r_2 - r_1)\frac{e^{ik|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|}\frac{e^{3ik|\vec{r}_1-\vec{r}_2|}}{|\vec{r}_1-\vec{r}_2|^3}\frac{e^{ik|\vec{r}-\vec{r}_2-\vec{x}|}}{|\vec{r}-\vec{r}_2-\vec{x}|}, \quad k \approx p_F/\hbar.$$

I am particularly interested in diagrams with "crossings" because external coordinates/momenta are involved in the internal integrations. I want to prove that this "crossing" term becomes important if p is not close to $p_F/\hbar$. It is better to calculate it in the momentum representation where we have one 3D-integrals less:

$$\int \frac{d^3q_1}{(2\pi)^3} \int \frac{d^3q_2}{(2\pi)^3} \frac{1}{E-(\vec{p}-\vec{q}_1)^2/(2m)+i0} \cdot \frac{1}{E-(\vec{p}-\vec{q}_1-\vec{q}_2)^2/(2m)+i0} \cdot \frac{1}{E-(\vec{p}-\vec{q}_2)^2/(2m)+i0}.$$

First, I am not sure that this integral converges. Suppose $p \approx p_F/\hbar$; then it seems that the characteristic scale of integration is $q_{1,2} \sim p - p_F/\hbar$. Now let us consider small values of $p \ll p_F/\hbar$. Then the characteristic integration scale grows. I mean, the integral grows when p decreases. I imagine that for small values of p its imaginary part may be not smaller than $1/\tau$. If this is correct, then for values of p far from $p_F/\hbar$ it is insufficient to take only "ladder"-like (i.e., "rainbow"-like) diagrams in the calculation of τ; diagrams with intersecting disorder lines may give important p-dependent contribution. (Where p is the *external* momentum.)

This Section has been born in intensive discussions with D. Maslov.

Chapter 2

Average of two Green's functions

STOP Conflicting notations: D sometimes stands for the diffuson self energy, and sometimes – for the diffusion coefficient.

STOP Развить такую идею: То, что у нас не возникает моментов (средних) выше второго порядка, есть следствие предположения о Гауссовости нашего распределения (1.4). А то, что распределение -Гауссово, сказано, к примеру в работе A. I. Larkin and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. **91**, 1815 (1986) [Sov. Phys. JETP **64**, 1075 (1986)], а также в ([cond-mat/9810191]10). We consider only "ladder" diagrams, which give the main contribution to the results (see Sec. 3.7).

STOP Using (13.5), we have (in case when $A = 0$)

$$C_{p_2}^{p_1} = \frac{\tilde{\tau}}{\tau_0\tau_\epsilon}\frac{G_R(p_1)G_A(p_2)}{-i(E_1-E_2)+D_\epsilon(\vec{p}_1+\vec{p}_2)^2}, \quad D_{p_2}^{p_1} = \frac{\tilde{\tau}}{\tau_0\tau_\epsilon}\frac{G_R(p_1)G_A(p_2)}{-i(E_1-E_2)+D_\epsilon(\vec{p}_1-\vec{p}_2)^2}, \quad \epsilon \equiv \frac{E_1+E_2}{2}. \tag{2.1}$$

In the opposite case, $lq \gg 1$, we have [expanding (13.2) and **ignoring the ξ-dependence of DoS** ν] in 2D

$$\frac{C_{\vec{q}-\vec{p},E-\omega}^{\vec{p},E}}{G_R(\vec{p},E)G_A(\vec{q}-\vec{p},E-\omega)} = \frac{D_{\vec{p}-\vec{q},E-\omega}^{\vec{p},E}}{G_R(\vec{p},E)G_A(\vec{p}-\vec{q},E-\omega)} = 1+\frac{1}{lq}+\frac{1}{l^2q^2}+\frac{1+2i\omega\tau}{2l^3q^3}, \tag{2.2}$$

and in 3D

$$\frac{C_{\vec{q}-\vec{p},E-\omega}^{\vec{p},E}}{G_R(\vec{p},E)G_A(\vec{q}-\vec{p},E-\omega)} = \frac{D_{\vec{p}-\vec{q},E-\omega}^{\vec{p},E}}{G_R(\vec{p},E)G_A(\vec{p}-\vec{q},E-\omega)} = 1+\frac{\pi}{2lq}+\frac{\pi^2/4-1+i\omega\tau}{l^2q^2},$$

We can see that in this case cooperon and diffuson are like in the ballistic case (when there are no impurities). This is because of that if we are interested in very small space scales, the concentration of impurities becomes effectively very small.

2.1 How $\vec{A} = \text{const}$ affects cooperon and diffuson

The influence of the magnetic field can be studied on 3 complexity levels: (i) when it only manifests itself as a phase shift (constant vector potential, e.g. in a mesoscopic ring), (ii) due to the term $-\mu\vec{\sigma}\vec{B}$ and (iii) making electrons precessing, due to the appearance of $\vec{A}$ in the term $(p - e\vec{A}/c)^2/(2m)$ in Hamiltonian.

Intuitive consideration: We know that application of $\vec{A}$ is equivalent to adding the term $-e\vec{A}/c$ to momentum operators (see e.g. ([27]4.41)). Due to the fact that the denominator of a diffuson depends only on the difference between momenta, we may expect that the diffuson will not depend on the applied field (except for the $\vec{A}$-dependence of GFs in the numerator).

Here we consider the simplest case when the vector potential $\vec{A}$ is constant both in time and space. The calculations are performed for the case of a cooperon; afterwards it will be evident that the diffuson is not affected by $\vec{A}$.

In order to get the perturbed self energy we can substitute G_R and G_A in

$$= \Sigma_2^{(0)} = \frac{1}{(2\pi\nu_0\tau)^2}\int\frac{d^dp}{(2\pi)^d}G_R(p,E)G_A(k-p,E-\omega)$$

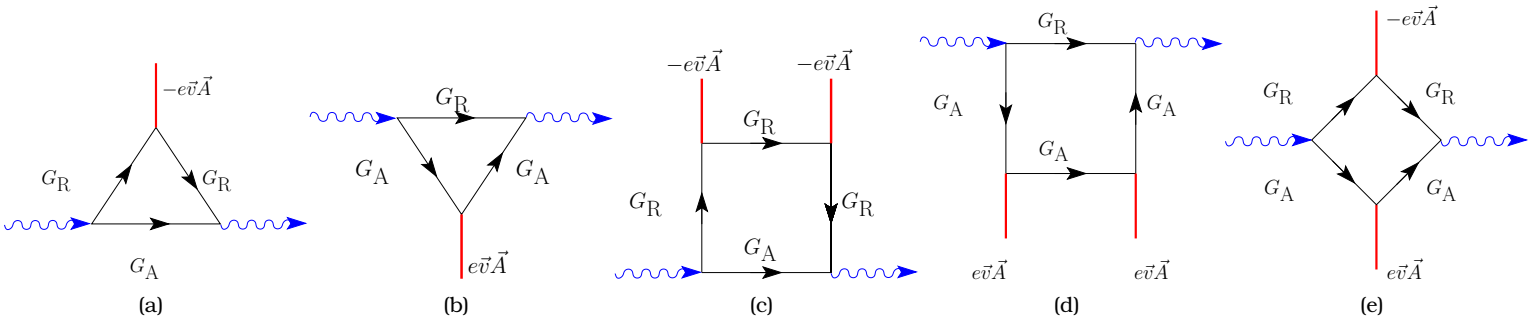


Figure 2.1: Diagrams for calculating self energy Σ_2 of a cooperon in case when $\vec{A} \neq 0$, $\vec{A} = \text{const}$. In (b) and (e) the change of sign in the vertex (3.9) occurs because the momentum argument of $G_A(\vec{k}-\vec{p})$ is $\vec{k}-\vec{p} \approx -\vec{n}p_F$. This is not true for the diffuson case.

α	1	1	2	2	2
β	0	1	0	1	2
diagram from Fig.2.1	b	a	d	e	c

Table 2.1: The correspondence between diagrams in Fig.2.1 and values of α and β in ([28]A2.3). (see `LEARN/dephasing.tex`)

with the expressions for them according to the perturbation theory (where the perturbation is the applied field $\vec{A}$).

Note[1] that as an unperturbed self energy we must take not $\Sigma_2^{(0)}$ (which is reducible), but the irreducible part of it which is $\Sigma_1 = (2\pi\nu_0\tilde{\tau})^{-1}$.

The first-order perturbation $\Sigma_2^{(1)}$ is represented by two "triangular" diagrams 2.1(a) and 2.1(b) which are both equal to an expression (omitting $n^2U_0^4$ coefficients):

$$\frac{1}{2\pi\nu_0\tilde{\tau}}\int\frac{\mathrm{d}^d p}{(2\pi)^d}(-e\vec{A},\vec{v})G_R^2(\vec{p},E)G_A(\vec{k}-\vec{p},E-\omega) == \frac{1}{2\pi\nu_0\tilde{\tau}}\int\frac{\mathrm{d}^d p}{(2\pi)^d}(-e\vec{A},-\vec{v})G_R(\vec{p},E)G_A^2(\vec{k}-\vec{p},E-\omega) = -2D_\epsilon\tau_\epsilon\frac{\tau_0}{\tau}\left(-e\vec{A}\vec{k}\right).$$

The second-order correction $\Sigma_2^{(2)}$ is represented by three diagrams [figs. 2.1(c), 2.1(d) and 2.1(e)]:

$$\begin{aligned}\frac{1}{2\pi\nu_0\tilde{\tau}}\int\frac{\mathrm{d}^d p}{(2\pi)^d}(-e\vec{A},\vec{v})^2G_R^3(\vec{p},E)G_A(\vec{k}-\vec{p},E-\omega) &= \frac{1}{2\pi\nu_0\tilde{\tau}}\int\frac{\mathrm{d}^d p}{(2\pi)^d}(-e\vec{A},\vec{v})^2G_R(\vec{p},E)G_A^3(\vec{k}-\vec{p},E-\omega)\\ &= -\frac{1}{2}\frac{1}{2\pi\nu_0\tilde{\tau}}\int\frac{\mathrm{d}^d p}{(2\pi)^d}(-e\vec{A},\vec{v})^2G_R^2(\vec{p},E)G_A^2(\vec{k}-\vec{p},E-\omega) = -D_\epsilon\tau_\epsilon\frac{\tau_0}{\tilde{\tau}}\left(-e\vec{A}\vec{k}\right)^2.\end{aligned}$$

Finally we obtain that cooperon changes in the following way: instead of $(2\pi\nu_0\tilde{\tau})^{-1}$ we have

$$\frac{1}{2\pi\nu_0\tilde{\tau}}\left[1-D_\epsilon\tau_\epsilon\frac{\tau_0}{\tilde{\tau}}\left(-4e\vec{A}\vec{k}+4e^2A^2\right)\right]. \tag{2.3}$$

So, in accordance with the intuitive consideration, when $\vec{A} \neq 0$ we have (compare with (2.1))

$$C^{\vec{p},E}_{\vec{k}-\vec{p},E-\omega} = \frac{\tilde{\tau}}{\tau_0}\frac{\tilde{G_R}(\vec{p},E)\tilde{G_A}(\vec{k}-\vec{p},E-\omega)}{l_\epsilon^2(\vec{k}-2e\vec{A})^2/d - i\omega\tau_\epsilon}, \tag{2.4}$$

where $\tilde{G}$ denotes Green functions changed due to the applied $\vec{A}$.

For the diffuson we would have the same consequence of formulas with $\vec{q}-\vec{p}$ substituted by $\vec{p}-\vec{q}$. It is clear that such substitution would lead to cancellation of all the considered diagrams (see Fig. 2.1): the first two diagrams cancel each other, while the 3rd and the 4th cancel the 5th one.

[1] More profound proof: $C = C^{(0)} + C^{(1)} + C^{(0)}\Sigma_2 C$, where $C^1 = C^0\Sigma_1 C^0$ and $\Sigma_2 = \Sigma_2^{(0)} + \Sigma_2^{(1)} + \Sigma_2^{(2)}$. From here it is evident that $\Sigma_1 C^{(0)} + \Sigma_2^{(0)} C = \Sigma_1 C$, so we arrive to our statement: $C = C^{(0)} + C^{(0)}\left(\Sigma_1 + \Sigma_2^{(1)} + \Sigma_2^{(2)}\right)C$.

An important note: from (2.4) one sees that cooperon self energy (see sec. 2.2) in coordinate space is

$$\frac{1}{2\pi\nu_0\tau_0\tau_\epsilon}\frac{1}{V}\sum_{\vec{n}\in\mathbb{Z}^d}\frac{e^{i\vec{k}_{\vec{n}}\vec{r}}}{D_\epsilon(\vec{k}_{\vec{n}}-2e\vec{A})^2-i\omega},\quad \vec{k}_{\vec{n}}=\frac{2\pi\vec{n}}{\vec{L}}. \tag{2.5}$$

On the other hand, from (13.10) we have:

$$\frac{1}{2\pi\nu_0\tau_0\tau_\epsilon}\frac{1}{V}\sum_{\vec{n}\in\mathbb{Z}^d}\frac{e^{i\left(\vec{k}_{\vec{n}}+2e\vec{A}\right)\vec{r}}}{D_\epsilon k_{\vec{n}}^2-i\omega},\quad \vec{k}_{\vec{n}}=\frac{2\pi\vec{n}}{\vec{L}}-2e\vec{A}. \tag{2.6}$$

These two expressions are equivalent. So one can introduce a rule for writing expressions for the diagrams in momentum representation in case $\vec{A}=\text{const}$: write the same as if $A=0$ but change all sums over the momentum from (2.5) to (2.6). The form (2.6) seems to be more useful because its definition of $\vec{k}_{\vec{n}}$ is the same as for (13.10).

2.2 Considering self energy

См., напр., spinHall.pdf. One can calculate more general objects: $\mathfrak{C}^{p_1p_3}_{p_2p_4}\equiv\overline{G_\mathrm{R}(p_1,p_3)G_\mathrm{A}(p_2,p_4)}$ and $\mathfrak{D}^{p_1p_3}_{p_2p_4}\equiv\overline{G_\mathrm{R}(p_1,p_3)G_\mathrm{A}(p_4,p_2)}$. Only "ladder" diagrams are taken into account here. Due to this fact $\mathfrak{D}^{p_1p_3}_{p_2p_4}\neq\mathfrak{C}^{p_1p_3}_{p_4p_2}$ (even when $\vec{A}=0$).

Here are the key points for the calculation of self energy:

1. Every wavy line due to the averaging produces δ-function representing momentum conservation laws.
2. Integration is taken over momenta of all GFs, which are between the wavy lines. Finally we have the integration over momentums which are carried by wavy lines and one δ function, representing total momentum conservation in the diagram.
3. Except for the case $r=0$ diagram with $r+1$ wavy lines can be obtained from the one with r wavy lines by multiplying by independent of r magnitude $X^{(\pm)}$. Here it is essential that in the considered model the impurity potential is constant in momentum space: $U(\vec{q})=U_0$.

After all we get[2]

$$\mathfrak{C}^{p_1p_3}_{p_2p_4}=(2\pi)^{d+1}\delta(p_1+p_2-(p_3+p_4))G_\mathrm{R}(p_1)G_\mathrm{A}(p_2)\times\left[(2\pi)^{d+1}\delta(p_1-p_3)+\frac{1}{2\pi\nu_0\tilde{\tau}}\cdot\frac{G_\mathrm{R}(p_3)G_\mathrm{A}(p_4)2\pi\delta(E_1-E_3)}{1-X^{(+)}}\right], \tag{2.7}$$

$$\mathfrak{D}^{p_1p_3}_{p_2p_4}=(2\pi)^{d+1}\delta(p_1+p_4-(p_3+p_2))G_\mathrm{R}(p_1)G_\mathrm{A}(p_2)\times\left[(2\pi)^{d+1}\delta(p_1-p_3)+\frac{1}{2\pi\nu_0\tilde{\tau}}\cdot\frac{G_\mathrm{R}(p_3)G_\mathrm{A}(p_4)2\pi\delta(E_1-E_3)}{1-X^{(-)}}\right]. \tag{2.8}$$

For $\vec{A}=0$ one can obtain (2.1) from (2.7) and (2.8) by integrating over p_3 and p_4. In case of $\vec{A}\neq0$ this correspondence is approximate; the exact one would be if one expands $G_\mathrm{R}(p_3)G_\mathrm{A}(p_4)$ in (2.7) in perturbation series up to the second order by $\vec{A}$, like in sec. 2.1.

So, the question arises: must we change the Green functions in numerators of (2.7) and (2.4) or not and (if must) how? An expansion up to the second order by $\vec{A}$ would provide the exact correspondence between (2.1) and (2.7), while the use of exact formula (13.10) would provide periodicity in $\vec{A}$ in cylindrical geometry. The latter seem to work, see sec. 8.12.

So, we have[3]

$$\overline{G_\mathrm{R}(p_1,p_3)G_\mathrm{A}(p_2,p_4)}=\left[\mathfrak{C}^{p_1p_3}_{p_2p_4}+\mathfrak{D}^{p_1p_3}_{p_4p_2}-\mathfrak{C}^{(0)}-\mathfrak{C}^{(1)}\right]\left[1+O\left(\frac{\hbar}{p_\mathrm{F}l}\right)\right], \tag{2.9}$$

where $\mathfrak{C}^{(0)}=\mathfrak{D}^{(0)}$ and $\mathfrak{C}^{(1)}=\mathfrak{D}^{(1)}$ are the zeroth and first order diagrams for cooperon (or diffuson). We can also consider somewhat "universal" self energy:

$$C^{p_1p_3}_{p_2p_4}=\frac{1}{2\pi\nu_0\tau_0\tau_\epsilon}\frac{(2\pi)^{d+2}\delta(p_1+p_2-(p_3+p_4))\delta(E_1-E_3)}{-i(E_1-E_2)+D_\epsilon\left(\vec{p}_1+\vec{p}_2-2e\vec{A}\right)^2}, \tag{2.10}$$

$$D^{p_1p_3}_{p_2p_4}=\frac{1}{2\pi\nu_0\tau_0\tau_\epsilon}\frac{(2\pi)^{d+2}\delta(p_1-p_2-(p_3-p_4))\delta(E_1-E_3)}{-i(E_1-E_2)+D_\epsilon(\vec{p}_1-\vec{p}_2)^2}, \tag{2.11}$$

[2] I wrote the formulas in this subsection for more common case with the applied constant vector potential $\vec{A}$, see sec. 2.1.

[3] Or may be, instead of subtracting $C^{(1)}$ in (2.9) it is better just to say that, by definition, a cooperon has no diagram with one impurity line. I think this is like I did in sec. 7.

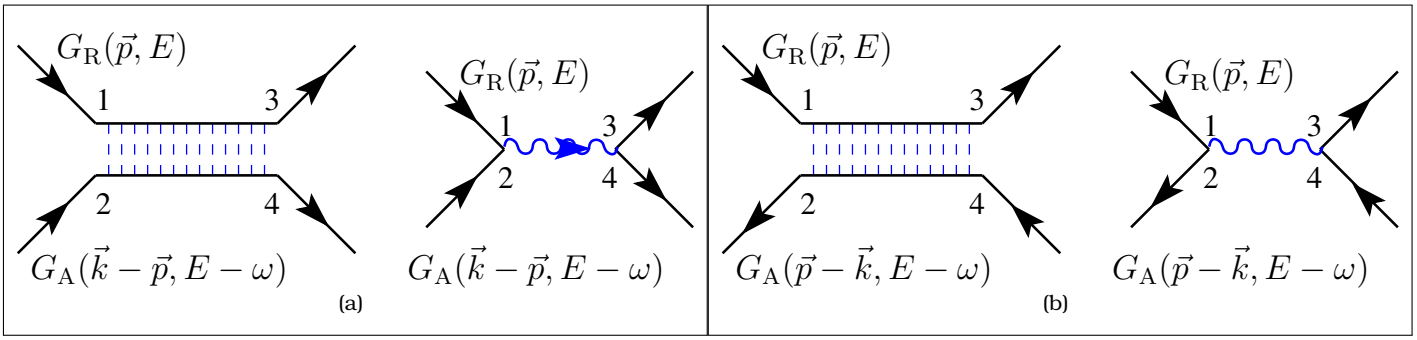


Figure 2.2: Diagrams for (a) cooperon and (b) diffuson.

where D_ϵ is the diffusion coefficient. Notice that (2.10) and (2.11) differ from their values in case of constant DoS only by substitution of τ, D with their energy-dependent generalizations τ_ϵ, D_ϵ, etc.
However, I doubt that this will be so beautiful for Hikami boxes.

From this point until the end of the section **the ξ - dependence of ν is ignored**.

There is no energy transfer in cooperon or diffuson lines, so in energy representation we have simple products of the type: $G_R(E_1)G_R(E_1)\Sigma_{E_2}^{E_1}G_A(E_2)G_A(E_2)$. In time representation we apparently have complication: we get convolution (свёртку). Let us write the expressions for C and D in momentum- time representation (with the removed $(2\pi)^2\delta(E_1-E_3)\delta(E_2-E_4)$):

$$C_{\vec{p}_2\vec{p}_4;t_2}^{\vec{p}_1\vec{p}_3;t_1} = \frac{(2\pi)^d\delta(\vec{p}_1+\vec{p}_2-(\vec{p}_3+\vec{p}_4))\delta(t_1+t_2)\vartheta(t_1)\exp\left[-Dt_1\left(\vec{p}_1+\vec{p}_2-2e\vec{A}\right)^2\right]}{2\pi\nu_0\tau^2}; \tag{2.12}$$

$$D_{\vec{p}_2\vec{p}_4;t_2}^{\vec{p}_1\vec{p}_3;t_1} = \frac{(2\pi)^d\delta(\vec{p}_1+\vec{p}_4-(\vec{p}_3+\vec{p}_2))\delta(t_1+t_2)\vartheta(t_1)\exp\left[-Dt_1\left(\vec{p}_1-\vec{p}_2\right)^2\right]}{2\pi\nu_0\tau^2}; \tag{2.13}$$

In coordinate-energy representation the calculation is analogous to the GF in Sec. 1.5: in 3D we get exponential coordinate dependence, while in 2D we get Hankel function.

In coordinate-space representation for the self energies of cooperon and diffuson we get

$$C_{\vec{r}_2\vec{r}_4;t_2}^{\vec{r}_1\vec{r}_3;t_1} = \int\frac{\mathrm{d}^d p_{1\ldots4}}{(2\pi)^{4d}}e^{i\vec{p}_1\vec{r}_1-i\vec{p}_3\vec{r}_3}e^{i\vec{p}_2\vec{r}_2-i\vec{p}_4\vec{r}_4}\int\frac{\mathrm{d}E_1\mathrm{d}E_2}{(2\pi)^2}e^{-i(E_1t_1+E_2t_2)}C_{p_2p_4;E_2}^{p_1p_3;E_1}, \tag{2.14}$$

$$D_{\vec{r}_2\vec{r}_4;t_2}^{\vec{r}_1\vec{r}_3;t_1} = \int\frac{\mathrm{d}^d p_{1\ldots4}}{(2\pi)^{4d}}e^{i\vec{p}_1\vec{r}_1-i\vec{p}_3\vec{r}_3}e^{i\vec{p}_4\vec{r}_4-i\vec{p}_2\vec{r}_2}\int\frac{\mathrm{d}E_1\mathrm{d}E_2}{(2\pi)^2}e^{-i(E_1t_1+E_2t_2)}D_{p_2p_4;E_2}^{p_1p_3;E_1},$$

In 3D $$C_{\vec{r}_2\vec{r}_4;t_2}^{\vec{r}_1\vec{r}_3;t_1} = \delta(\vec{r}_1-\vec{r}_2)\delta(\vec{r}_3-\vec{r}_4)\delta(t_1+t_2)\vartheta(t_1)\frac{1}{2\pi\nu_0\tau^2}\left(\frac{\pi}{Dt_1}\right)^{d/2}\times\frac{1}{(2\pi)^d}\exp\left[-\frac{(\vec{r}_2-\vec{r}_4)^2d}{4l^2t_1/\tau}\right]\exp\left[2ie(\vec{r}_2-\vec{r}_4)\vec{A}\right], \tag{2.15}$$

$$D_{\vec{r}_2\vec{r}_4;t_2}^{\vec{r}_1\vec{r}_3;t_1} = \delta(\vec{r}_1-\vec{r}_2)\delta(\vec{r}_3-\vec{r}_4)\delta(t_1+t_2)\vartheta(t_1)\frac{1}{2\pi\nu_0\tau^2}\left(\frac{\pi}{Dt_1}\right)^{d/2}\times\frac{1}{(2\pi)^d}\exp\left[-\frac{(\vec{r}_2-\vec{r}_4)^2d}{4l^2t_1/\tau}\right]. \tag{2.16}$$

From (2.15) and (2.16) we see that $C_{\vec{r}_2\vec{r}_4;t_2}^{\vec{r}_1\vec{r}_3;t_1} = D_{\vec{r}_2\vec{r}_4;t_2}^{\vec{r}_1\vec{r}_3;t_1}$ for $\vec{A}=0$.

Because of δ-functions in (2.15) and in (2.16) in coordinate representation it is natural to draw the diagrams for C and D like in Fig. 2.2. In coordinate space we arrive to somewhat new diagram technique: a solid line denotes Green function with coordinates in the ends of the line; to a wavy line corresponds cooperon or diffuson self energy given by (2.15) or (2.16).

We are interested in processes, which have characteristic times $\gg\tau$. Here this means $t_1\gg\tau$; as a consequence, the characteristic space scale of a cooperon and diffuson is much bigger than l.

When studying in 2D case cylindrical (or torus) geometry with $A = A_x \neq 0$, we will have to study discrete version of cooperon self energy (2.15). From (13.13) it follows that

$$\begin{aligned}\frac{1}{L_x}\sum_{q_{nx}} \exp\left[i(q_{nx}-2eA)r_x - (q_{nx}-2eA)^2 Dt\right] &= \frac{1}{L_x}\sum_{n} \exp\left[i\frac{2\pi}{L_x}\left(n-\frac{\varphi}{\varphi_0}\right)r_x - \left(\frac{2\pi}{L_x}\right)^2\left(n-\frac{\varphi}{\varphi_0}\right)^2 Dt\right] \\ &= \clubsuit \frac{1}{2\sqrt{\pi D t}}\sum_{m\in\mathbb{Z}} e^{2\pi i m\Phi/\Phi_0}\exp\left[-\frac{(r_x - mL_x)^2}{4Dt}\right],\end{aligned} \tag{2.17}$$

where $\Phi = AL_x$ and $q_{nx} = 2\pi n/L_x;\ n \in \mathbb{Z}$.

---

the following is to be combined with the previous discussion.
From (13.13) it follows that

$$C^{\vec{x}_1\vec{x}_2;E_1}_{\vec{x}_7\vec{x}_8;E_2} = C^{\vec{x}_1\vec{x}_2;E_1}_{E_2}\sum_{n_1,n_2\in\mathbb{Z}} \delta\left(\frac{x_1-x_7}{2L}-n_1\right)\delta\left(\frac{x_2-x_8}{2L}-n_2\right) = C^{\vec{x}_1\vec{x}_2;E_1}_{E_2}\delta(x_1-x_7)\delta(x_2-x_8),$$

if we restrict the coordinates to have values $0 \le x < L$.

Finally, let us write the Dyson equation for the diffuson in the mixed (Wigner) Rp-representation. In an inhomogeneous case

$$D(\vec{R},\vec{p}) = 1 + \exp\left[\frac{i}{2}\left(\partial^X_R\partial^D_p - \partial^X_p\partial^D_R\right)\right]X(\vec{R},\vec{p})D(\vec{R},\vec{p}), \tag{2.18}$$

where $X(\vec{R},\vec{p})$ is the diffuson "bubble" in the Wigner representation. Dyson equation for cooperon is analogous to (2.18).

Chapter 3

Charge conductivity: simplest case

I probably should erase most of this section, since I've published a good (and short) explanation in [20]. The general relation between j, E, and σ for the system without spatial dispersion:

$$j(t) = \int_0^\infty \sigma(t_1)E(t-t_1)\mathrm{d}t_1 \equiv \int_{-\infty}^\infty \sigma(t_1)E(t-t_1)\mathrm{d}t_1, \quad \sigma(t_1<0) \stackrel{\mathrm{df}}{=} 0 \Longrightarrow j(\omega) = \sigma(\omega)E(\omega) = \sigma(\omega)\frac{i\omega}{c}A_\omega. \tag{3.1}$$

In the important case, when a problem with initial condition is considered, $E(t<0) \equiv 0$ and $j(t<0) \equiv 0$, so that (3.1) is valid also for Laplace transforms (which we denote with tilde):

$$\tilde{j}(-i\omega) = \tilde{\sigma}(-i\omega)\tilde{E}(-i\omega) \Longrightarrow \tilde{j}(\omega) = \tilde{\sigma}(\omega)\tilde{E}(\omega),$$

provided that functions $\tilde{j}(p)$, $\tilde{E}(p)$, and $\tilde{\sigma}(p)$ have no branchcuts. See also (13.7).

See p.[29]100-102 and p.[29]115-116 for the connection between conductivity (проводимость) and conductance (электропроводность).

Below the conductivity is calculated using two techniques: Matsubara and the one for $T = 0$. However, both of them are questionable in this (non-equilibrium!) case.

The Matsubara technique was designed (and guaranteed to give correct results) only for the equilibrium processes, when the Hamiltonian is time-independent. So what is written in section 3.2 should be considered as a recipe that works, nothing more. The ground of the method lies in Keldysh technique - see Sec. 9. As for the $T = 0$ technique, in addition it has a disadvantage that averaging in it is performed on the ground state only, so that all the excited states are not taken in account.

At first I will calculate the major (Drude) contribution to the conductivity, which corresponds to the "bubble" diagram in Fig. 3.1(a). This is a sort of a "warm-up" which helped me to understand different diagrammatic techniques: zero-temperature, Matsubara (both described in [5]), and Keldysh (described in [21] and in Sec. 9). In Sec. 3.4 I calculate the WL correction using Matsubara[1].

3.1 Calculations using zero-temperature technique

To study conductivity we need expressions for the operator of current density (in the interaction representation) and its value $\vec{j}(x) = \langle 0|\hat{\vec{j}}(x)|0\rangle$ [compare with (13.9)]

$$\hat{\vec{j}}(x) = \frac{ie\hbar}{2m}\left[(\nabla\hat{\psi}^\dagger(x))\hat{\psi}(x) - \hat{\psi}^\dagger(x)\nabla\hat{\psi}(x)\right] - \frac{e^2}{mc}\vec{A}(x)\hat{\psi}^\dagger(x)\hat{\psi}(x); \quad x \equiv (\vec{r},t), \tag{3.2}$$

where the first term called "kinetic part", and the second — "diamagnetic part" see, e.g., p.[30]160.

$$\vec{j}(x) = \frac{e\hbar}{2m}\lim_{x'\to x+0}(\nabla_{r'} - \nabla_r)\,G(x,x') - \frac{ne^2}{mc}\vec{A}(x). \tag{3.3}$$

[1] If I knew wonderful Keldysh at the time when I wrote this section, I would certainly use it instead of disgusting Matsubara. It is not only my opinion, see cond-mat/9810191, 0109316.

We wrote (3.3) to first order in $\vec{A}(x)$, so the dependence of $N = \langle 0|\psi^\dagger(x)\psi(x)|0\rangle$ on x was neglected. In the Hamiltonian calibration (scalar potential $\equiv 0$, see ([5]29.1))), our perturbation operator is given by $\hat{V}(t) = -\frac{1}{c}\int \vec{A}(y)\hat{\vec{j}}(y)\mathrm{d}^d y$. To first order in $\vec{A}$ we thus have

$$\hat{V}(t) = -\frac{ie\hbar}{2mc}\int \mathrm{d}^d y\vec{A}(y)\left[(\nabla\psi^\dagger(y))\psi(y) - \psi^\dagger(y)(\nabla\psi(y))\right]. \tag{3.4}$$

Then we utilize formulas ([5]8.7-9) and obtain (see I, 67):

$$\delta G^{(1)}(x,x') = -\frac{ie}{2mc}\int \mathrm{d}^{d+1} y\vec{A}(y)\lim_{y'\to y}\left[(\nabla_{\vec{y}'} - \nabla_{\vec{y}})G(x,y')G(y,x')\right], \tag{3.5}$$

$$\delta G^{(1)}(\vec{x},E;\vec{x}',E') = -\frac{ie}{2mc}\int \mathrm{d}^d y\vec{A}(\vec{y},E-E')\lim_{\vec{y}'\to\vec{y}}\left[(\nabla_{\vec{y}'} - \nabla_{\vec{y}})G^E(\vec{x},\vec{y}')G^{E'}(\vec{y},\vec{x}')\right],$$

From (3.5) we have:

$$\delta G^{(1)}(p,p') = -\frac{e}{2mc}\int \frac{\mathrm{d}^{d+1}p_3\mathrm{d}^{d+1}p_4}{(2\pi)^{2(d+1)}}G(p,p_3)G(p_4,p')\vec{A}(p_3-p_4)(\vec{p}_3+\vec{p}_4) \tag{3.6}$$

- for the case when there is no spatial homogeneity, which reduces to

$$\delta G^{(1)}(p,p') = -\frac{e}{2mc}G(p)G(p')\vec{A}(p-p')(\vec{p}+\vec{p}') \tag{3.7}$$

in the homogeneous case. In the essential case of spatially homogeneous applied field $\vec{A}(x)$ we have $\vec{A}(q) = \vec{A}_\omega(2\pi)^d\delta(\vec{q})$ and (3.7) reduces to

$$\delta G^{(1)}(p,p') = -\frac{e}{c}G(p)G(p')\vec{v}\vec{A}(E-E')(2\pi)^d\delta(\vec{p}-\vec{p}'). \tag{3.8}$$

But on the other hand we know that $\delta G^{(1)}(p,p') = G(p)V(p-p')G(p')/\hbar$, so we see that in this case

$$\text{the electromagnetic vertex}\quad V(q) = -\frac{e\hbar}{c}\vec{v}\vec{A}(q). \tag{3.9}$$

From (3.5) it follows that

$$\vec{j}(x) + \frac{ne^2}{mc}\vec{A}(x) = -\frac{ie^2\hbar}{4m^2c}\lim_{x'\to x}\left\{(\nabla_{\vec{r}'} - \nabla_{\vec{r}})\int \mathrm{d}^{d+1} y\vec{A}(y)\lim_{y'\to y}\left[(\nabla_{\vec{y}'} - \nabla_{\vec{y}})G(x,y')G(y,x')\right]\right\}, \tag{3.10}$$

which is almost[2] the formula ([5]39.1).

Let's make Fourier transformation of (3.5) and of (3.10):

$$G(x,y') = \int \frac{\mathrm{d}^{d+1}p_1\mathrm{d}^{d+1}p_3}{(2\pi)^{2(d+1)}}G(p_1,p_3)e^{i(p_1x-p_3y')},\quad \vec{A}(y) = \int \frac{\mathrm{d}^{d+1}q}{(2\pi)^{d+1}}e^{iqy}\vec{A}(q),$$

$$G(y,x') = \int \frac{\mathrm{d}^{d+1}p_2\mathrm{d}^{d+1}p_4}{(2\pi)^{2(d+1)}}G(p_4,p_2)e^{i(p_4y-p_2x')} = \int \frac{\mathrm{d}^{d+1}p_2\mathrm{d}^{d+1}p_4}{(2\pi)^{2(d+1)}}G(-p_4,-p_2)e^{i(p_2x'-p_4y)}.$$

In Fourier-space (3.10) is equivalent to

$$\vec{j}(q) + \frac{ne^2}{mc}\vec{A}(q) = \frac{ie^2\hbar}{4m^2c}\int \frac{\mathrm{d}^{d+1}p_1}{(2\pi)^{d+1}}\frac{\mathrm{d}^{d+1}p_3\mathrm{d}^{d+1}p_4}{(2\pi)^{2(d+1)}}G(p_1,p_3)G(p_4,p_1-q)\times$$
$$(2\vec{p}_1-\vec{q})\left[\vec{A}(p_3-p_4)(\vec{p}_3+\vec{p}_4)\right]. \tag{3.11}$$

Under the integral we have two Green's functions that are to be averaged. It is clear that the integral in (3.11) is nonzero only if $\vec{p}_1$ and $\vec{p}_3$ are correlated; particularly in the case when $\vec{p}_1 = \vec{p}_3$ which occurs in zeroth order of averaging[3], i.e. when $\langle GG\rangle = \langle G\rangle\langle G\rangle$. Keeping this in mind[4], we consider only the zeroth order and thus have

$$\vec{j}(q) + \frac{ne^2}{mc}\vec{A}(q) = \frac{ie^2\hbar}{4m^2c}\int \frac{\mathrm{d}^{d+1}p}{(2\pi)^{d+1}}G(p)G(p-q)(2\vec{p}-\vec{q})\vec{A}(q)(2\vec{p}-\vec{q}). \tag{3.12}$$

[2] Note that (3.10) is not consistent with [5] - in formula ([5]39.1) there stands 1/2.

[3] In other words, we have no diffuson here. Life becomes more complicated when impurity scattering or GF are not isotropic, and additional contributions, like diagram in Fig. 7.1(b) appear.

[4] This is not true when $\vec{q} \neq 0$. See also section 3.4.

Considering then the case of spatially homogeneous applied field, we have $q = (\omega, \vec{0}) \equiv \omega$ and[5]

$$\vec{j}(\omega) + \frac{ne^2}{mc}\vec{A}_\omega = \frac{ie^2\hbar}{m^2c}\int \frac{\mathrm{d}^{d+1}p}{(2\pi)^{d+1}} G(p)G(p-\omega)\vec{p}\vec{A}_\omega\vec{p}. \tag{3.13}$$

From the relation

$$\int \frac{\mathrm{d}^{d+1}p}{(2\pi)^{d+1}} G(\vec{p},E)G(\vec{p},E\pm\omega)p_ip_j \approx \frac{\nu_0\tau}{d}\frac{p_F^2}{\hbar^2}\delta_{ij}(|\omega| + i\omega^2\tau),$$

neglecting diamagnetic term $\frac{Ne^2}{mc}\vec{A}_\omega$, we have[6]

$$j_k(q) = \sum_l \frac{i\omega}{c}\sigma_{kl}A_l(q), \quad \sigma_{ij} = \sigma\delta_{ij}\,\mathrm{sign}\,\omega, \quad \sigma = \frac{ne^2\tau}{m\hbar} = \frac{e^2}{\hbar}\nu_0 D_0, \quad \frac{n}{m} = \frac{\nu_0 p_F^2}{m^2 d}. \tag{3.14}$$

σ_{ij} depends on $\mathrm{sign}\,\omega$ and thus is incorrect. Apart from this "Kleinigkeit" σ is correct, cf. Eq. ([21]4.7).

This problem may occur from the before mentioned illegacy of this technique in calculating any response functions.

3.2 Calculations using Matsubara technique

STOP From this point insert $\hbar$ and c.

The formulas in this section to some extent repeat those of Sec. 3.1.

The current operator and its value are given by

$$\hat{\vec{j}}(x) = \frac{ie\hbar}{2m}\left[(\nabla\psi^\dagger(x))\psi(x) - \psi^\dagger(x)\nabla\psi(x)\right] - \frac{e^2}{mc}\vec{A}(x)\psi^\dagger(x)\psi(x), \tag{3.15}$$

$$\vec{j}(x) = \langle\hat{\vec{j}}(x)\rangle = \frac{ie}{2m}\left(\nabla_{r'} - \nabla_r\right)|_{x'\to x+0}\mathfrak{G}(x,x') - \frac{ne^2}{mc}\vec{A}(x) \tag{3.16}$$

The perturbation operator is given by STOP in the next eq. I've later changed sign which resulted in consequtive sign changes

$$\hat{V}(t) = -\frac{ie}{2mc}\int \mathrm{d}^d y\vec{A}(y)\left[(\nabla\psi^\dagger(y))\psi(y) - \psi^\dagger(y)(\nabla\psi(y))\right].$$

$$\delta\mathfrak{G}^{(1)}(x,x') = -\frac{ie}{2mc}\int \mathrm{d}^d y\int_0^{1/T}\mathrm{d}\tau_1\vec{A}(\vec{y},\tau_1)\lim_{y'\to y}\left[(\nabla_{\vec{y}'} - \nabla_{\vec{y}})\mathfrak{G}(\vec{r},\tau;\vec{y},\tau_1)\mathfrak{G}(\vec{y}',\tau_1;\vec{r}',\tau')\right],$$

Here I insert somewhat new formula that permits me to do the rest of calculations almost automatically (because they do not deal with time or energy variables, but only with spatial ones):

$$\delta\mathfrak{G}(\vec{r},\vec{r}';E_n,E_{n'}) = -\frac{ie}{2mc}\int \mathrm{d}^d yT^4\sum_{k,l}\vec{A}_{E_k-E_l}(\vec{y})\lim_{\vec{y}'\to\vec{y}}\left[(\nabla_{\vec{y}'} - \nabla_{\vec{y}})\mathfrak{G}(\vec{r},\vec{y};E_n,E_l)\mathfrak{G}(\vec{y}',\vec{r}';E_k,E_{n'})\right],$$

$$\delta\mathfrak{G}^{(1)}(\vec{p},\vec{p}';E_n,E_{n'}) = -\frac{e}{2mc}\int \frac{\mathrm{d}^d p_3\mathrm{d}^d p_4}{(2\pi)^{2d}}T^2\sum_{k,l}\mathfrak{G}(\vec{p},\vec{p}_3;E_n,E_l)\mathfrak{G}(\vec{p}_4,\vec{p}';E_k,E_{n'})\vec{A}_{E_k-E_l}(\vec{p}_3-\vec{p}_4)(\vec{p}_3+\vec{p}_4),$$

$$s\vec{j}(\vec{q},\omega_n) + \frac{Ne^2}{mc}\vec{A}(\vec{q},\omega_n) = \tag{3.17}$$

$$\frac{ie^2}{4m^2}T^3\sum_{n,k,l}\int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_3\mathrm{d}^d p_4}{(2\pi)^{2d}}\mathfrak{G}(\vec{p}_1,\vec{p}_3;E_n,E_l)\mathfrak{G}(\vec{p}_4,\vec{p}_1-\vec{q};E_k,E_l-\omega_n)\times \tag{3.18}$$

$$(2\vec{p}_1-\vec{q})\vec{A}_{E_k-E_l}(p_3-p_4)(\vec{p}_3+\vec{p}_4). \tag{3.19}$$

s

[5] this expression can be expressed graphically as closed diagram with v_i and v_j lines - see Fig. 3.1(a).

[6] There is also the number of particles missing in (3.14), see the note after (3.22).

What is the sense of $\vec{j}(\omega_n)$ and $\vec{A}(\omega_n)$? I can not clearly answer this question now. I only follow the recipe: to make analytical continuation to continuous frequencies and get real current and vector potential. VK has told that the proof is that this technique do coincide with the Keldysh one, see Sec. 9. Олег Евтушенко advised me to read in [31] about this.

After the averaging, assuming $\vec{q} = 0$, we get

$$\vec{j}(\omega_n) + \frac{Ne^2}{mc}\vec{A}(\omega_n) = \frac{ie^2}{m^2} T \int \frac{\mathrm{d}^d p}{(2\pi)^d} \sum_l \mathfrak{G}(\vec{p}, E_l)\mathfrak{G}(\vec{p}, E_l - \omega_n)\vec{p}\vec{A}(\omega_n)\vec{p} \tag{3.20}$$

And now we arrive to a key point of the procedure. We are to make analytical continuation of $\vec{j}(\omega_n)$, $\vec{A}(\omega_n)$ and $\mathfrak{G}$, to the Fourier-transformation of the current, vector potential and of time-dependent Green functions G_R or G_A correspondingly. For this purpose we use formulae ([5]17.25):

$$\mathfrak{G}(\omega_n) = G_\mathrm{R}(i\omega_n), \quad \omega_n > 0; \quad \mathfrak{G}(\omega_n) = G_\mathrm{A}(i\omega_n), \quad \omega_n < 0.$$

We assume that somewhat a-like relation is valid for $\vec{j}(\omega_n)$ and $\vec{A}(\omega_n)$: $\vec{j}(\omega_n) = \vec{j}(i\omega)$ and $\vec{A}(\omega_n) = \vec{A}(i\omega)$.

Keeping this in mind, we make analytical continuation (see `rzi.pdf`) of the sum in (3.20) and obtain[7]:

$$\begin{aligned} T\sum_l \mathfrak{G}(\vec{p}, E_l)\mathfrak{G}(\vec{p}, E_l - \omega_n) = \\ = \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{4\pi} G_\mathrm{R}(\vec{p}, z)G_\mathrm{A}(\vec{p}, z-\omega)\tanh\frac{z}{2T} - \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{4\pi} G_\mathrm{R}(\vec{p}, z)G_\mathrm{A}(\vec{p}, z-\omega)\tanh\frac{z-\omega}{2T} + \\ + \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{4\pi} G_\mathrm{R}(\vec{p}, z)G_\mathrm{R}(\vec{p}, z-\omega)\tanh\frac{z-\omega}{2T} - \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{4\pi} G_\mathrm{A}(\vec{p}, z)G_\mathrm{A}(\vec{p}, z-\omega)\tanh\frac{z}{2T}. \end{aligned} \tag{3.21}$$

For simplicity let us consider the case of $T = 0$. Also we are interested in the limiting case $\omega \to 0$.

We expect that two terms in (3.21) would give us Drude formula for conductivity, while the last two ones are to cancel diamagnetic term in (3.20).

Using (13.33) with analogy of (3.14), we get conductivity[8][9]:

$$\sigma_{ij} = \sigma_\mathrm{D}\delta_{ij}, \ \sigma_\mathrm{D} = \frac{e^2}{h}E_\mathrm{F}\tau\frac{2\pi\nu_0}{m} = \frac{e^2}{\hbar}\nu_0 D = \frac{Ne^2\tau}{m\hbar} = \frac{e}{\hbar}N\mu, \ N \overset{\mathrm{df}}{=} \frac{mv_F^2\nu_0}{d}, \ \sigma = \frac{e^2}{\hbar}\mathbb{G}, \ \mathbb{G} = \nu_0 DS/L \gg 1, \tag{3.22}$$

where (for the dilute electron gas) $\mu_e = e\tau/m$ is the mobility[10] (=подвижность), σ is the conductance, and $\mathbb{G}$ is so-called "dimensionless conductance". It is dimensionless $\forall d$, and people (VK) call it "large parameter of the diagrammatic expansion in the disorder averaging technique". [However, based on (3.41) I would rather use this name for g] According to ([0305478]3.10b), the condition $\mathbb{G} \gg 1$ in (3.22) is crucial for the validity of the SCBA.

The conductance is proportional to the electron concentration: $\sigma = |e|n\mu_e$, Naïvely applying Ohms law, we conclude that the conductance of a wire with the length l and crossection λ^{d-1} is $e^2/(hd)$. The quantity $e^2/h = (25813\Omega)^{-1}$ is called "the unit of conductance"; one can explicitly distinguish (=выделить) it in the expression for the linear-responce conductivity (8.7). Why don't I have n in (3.22)? Because of the inapropriate normalization, see the note after (13.30). In fact, (3.22) gives the conductance per electron. See also (13.34).

3.3 Cancellation of the diamagnetic term

(See also Sec. 3.5) The part of (3.20) [see also (3.21)], which was not taken into account in (3.22) is equal to (after analytical continuation and before the averaging)[11]

$$\begin{aligned} \frac{ne^2}{mc}\vec{A}_\omega = \frac{ie^2}{4m^2}\int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_3 \mathrm{d}^d p_4}{(2\pi)^{2d}}(2\vec{p}_1 - \vec{q})\left[\vec{A}(\vec{p}_3 - \vec{p}_4, \omega)(\vec{p}_3 + \vec{p}_4)\right]\times \\ \times \int \frac{\mathrm{d}z}{4\pi}\left\{G_\mathrm{R}(\vec{p}_1, \vec{p}_3, z)G_\mathrm{R}(\vec{p}_4, \vec{p}_1 - \vec{q}, z-\omega)\left(\tanh\frac{z-\omega}{2T} - 1\right) - G_\mathrm{A}(\vec{p}_1, \vec{p}_3, z)G_\mathrm{A}(\vec{p}_4, \vec{p}_1 - \vec{q}, z-\omega)\left(\tanh\frac{z}{2T} - 1\right)\right\}. \end{aligned} \tag{3.23}$$

[7] Note that everywhere the expressions like $\int \frac{\mathrm{d}^d p}{(2\pi)^d}$ must be considered as approximations of $\frac{1}{V}\sum_{\vec{p}}$.

[8] Since we did not took into acount spin, σ_D in (3.22) gives *conductivity per spin projection*.

[9] С проводимостью связано характерное время рассасывания объёмного заряда $\tau_M = \varepsilon/(4\pi\sigma)$ – максвелловское время релаксации.

[10] Mobility is useful in not-dilute semiconductors, see pp. [32]14,28. According to ([33]7.36), $\mu_e = |e|\tau/m$.

[11] It is clear that the inserted ones in (3.23) do not change it.

The aim thus is to prove (3.23). Let us make the proof for the simplest case of homogeneous field $\vec{A}$ so that $\vec{q}=0$.

$$\int \frac{\mathrm{d}^d p_1 \mathrm{d}^d p_3}{(2\pi)^{2d}} p_{1i} p_{3j} G_{\mathrm{R}}(\vec{p}_1,\vec{p}_3,z) G_{\mathrm{R}}(\vec{p}_3,\vec{p}_1,z-\omega) = \int \frac{\mathrm{d}^d p_1 \mathrm{d}^d p_3}{(2\pi)^{2d}} \langle \vec{p}_1|\hat{p}_i|\vec{p}_1\rangle\langle\vec{p}_1|\hat{G}_{\mathrm{R}}|\vec{p}_3\rangle\langle\vec{p}_3|\hat{p}_j|\vec{p}_3\rangle\langle\vec{p}_3|\hat{G}_{\mathrm{R}}|\vec{p}_1\rangle =$$
$$= \frac{1}{V^2}\sum_{\vec{p}_{1n},\vec{p}_{3m}} \langle \vec{p}_{1n}|\hat{p}_i|\vec{p}_{1n}\rangle\langle\vec{p}_{1n}|\hat{G}_{\mathrm{R}}|\vec{p}_3\rangle\langle\vec{p}_3|\hat{p}_j|\vec{p}_3\rangle\langle\vec{p}_3|\hat{G}_{\mathrm{R}}|\vec{p}_{1n}\rangle =$$
$$= \frac{1}{V^2}\sum_{\left(\substack{\vec{p}_{1n},\vec{p}_{3m}\\ \vec{p}'_{1n'},\vec{p}'_{3m'}}\right)} \langle \vec{p}_{1n}|\hat{p}_i|\vec{p}_{1n}\rangle\langle\vec{p}_{1n}|\hat{1}|\vec{p}'_{1n}\rangle\langle\vec{p}'_{1n}|\hat{G}_{\mathrm{R}}|\vec{p}_{3m}\rangle\langle\vec{p}_{3m}|\hat{p}_j|\vec{p}_{3m}\rangle\langle\vec{p}_{3m}|\hat{1}|\vec{p}'_{3m}\rangle\langle\vec{p}'_{3m}|\hat{G}_{\mathrm{R}}|\vec{p}_{1n}\rangle =$$
$$= \frac{1}{V^2}\operatorname{Sp}\left(\hat{p}_i\hat{1}\hat{G}_{\mathrm{R}}\hat{p}_j\hat{1}\hat{G}_{\mathrm{R}}\right) = \frac{1}{V^2}\operatorname{Sp}\left(\hat{p}_i\hat{G}_{\mathrm{R}}\hat{p}_j\hat{G}_{\mathrm{R}}\right), \tag{3.24}$$

where

$$\hat{G}^E_{\mathrm{R/A}} = \left[E-\hat{H}\pm i\delta\right]^{-1}, \quad \delta=+0, \quad |\vec{p}\rangle = \frac{1}{\sqrt{V}}e^{i\vec{p}\vec{r}}, \quad \langle\vec{p}|\vec{p}'\rangle = \delta_{\vec{p}\vec{p}'}. \tag{3.25}$$

$$\operatorname{Sp}(p_i G^E p_j G^{E-\omega}) \approx \operatorname{Sp}(p_i G p_j G) + \omega \operatorname{Sp}(p_i G p_j G^2), \quad G\equiv G^E. \tag{3.26}$$

Note that the term $\omega\operatorname{Sp}(p_iGp_jG^2)$ in (3.26) is a non-universal correction (see p. 30) which must be ignored. I mean, we must disregard it not only because it is small, but because we everywhere disregard non-universal terms. In order to generalize the results of this Sec., we perform the rest of the calculation for velocity-operators[12] $\hat{\vec{v}}$. Then

$$v_i = -\frac{i}{\hbar}\left[x_i,\hat{H}\right] = \frac{i}{\hbar}\left[x_i, E-\hat{H}\pm\frac{i}{2\tau}\right] = \frac{i}{\hbar}\left[x_i, G^{-1}\right],$$
$$\operatorname{Sp}\left[v_iGv_jG\right] = \frac{i}{\hbar}\operatorname{Sp}\left[\left(x_iG^{-1}-G^{-1}x_i\right)Gv_jG\right] = \frac{i}{\hbar}\operatorname{Sp}([x_i,v_j]G) = -\delta_{ij}\operatorname{Sp}G, \tag{3.27}$$

where we've assumed that $[\hat{x}_i,\hat{p}_j]=[\hat{x}_i,m\hat{v}_j]$, which is true without SOI and for the linear SOI Hamiltonian (6.1).

$$\operatorname{Sp}(p_iGp_jG^2) = \frac{im}{\hbar}\cdot\operatorname{Sp}\left[\left(x_iG^{-1}-G^{-1}x_i\right)Gp_jG^2\right] = \frac{im}{\hbar}\left[-\operatorname{Sp}\left(x_ip_j\frac{\partial}{\partial E}G\right)-\operatorname{Sp}(x_iGp_jG)\right],$$
$$\operatorname{Sp}(x_iGp_jG) = \operatorname{Sp}([x_i,x_j]G) = 0. \tag{3.28}$$

Eq. (3.28) is important for the derivation of the Kubo-Greenwood formula for the conductivity, which is exact in the limit of small ω, see eq. (2.4) from PRB**48**15218.

For G_{A} we would have the same. The proof is made for the case of $\omega=0$. From Lehmann representation[13]

$$G_{\mathrm{R/A}}(\vec{r},\vec{r}';E) = \sum_n \psi_n(\vec{r})\psi_n^*(\vec{r}')E-E_n\pm i\varepsilon, \quad \varepsilon=+0, \tag{3.29}$$

we see that in case of no interaction between the electrons we have (cf. pp. [25]28-37)

$$G_{\mathrm{R}}(\vec{r},\vec{r}';E)-G_{\mathrm{A}}(\vec{r},\vec{r}';E) = -2\pi i\sum_n \delta(E-E_n)\psi_n(\vec{r})\psi_n^*(\vec{r}'), \tag{3.30}$$

where the sum is taken over all *distinct* states (some of the states may be degenerate), so that

$$\frac{1}{V}\int \mathrm{d}^d r\left[G_{\mathrm{R}}(\vec{r},\vec{r};E)-G_{\mathrm{A}}(\vec{r},\vec{r};E)\right] = -2\pi i\nu_E V, \tag{3.31}$$

where ν_E is the density of states defined by (1.13). Note that (for a homogeneous system) coordinate integration here is fictitious, because integrand does not depend on coordinates. (In fact, it only gives V)

From (3.31) one obtains:

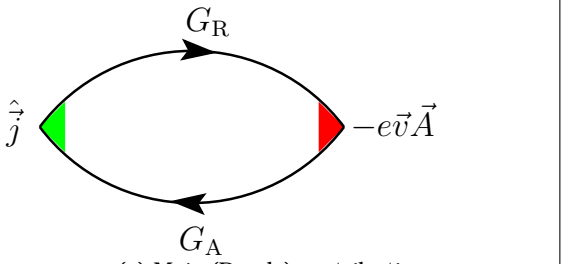
$$\frac{1}{V}\sum_p\left[G_{\mathrm{R}}(\vec{p},E)-G_{\mathrm{A}}(\vec{p},E)\right] = -2\pi i\nu_E V. \tag{3.32}$$

Supposing that disorder averaged Green function has the form (1.9) one obtains

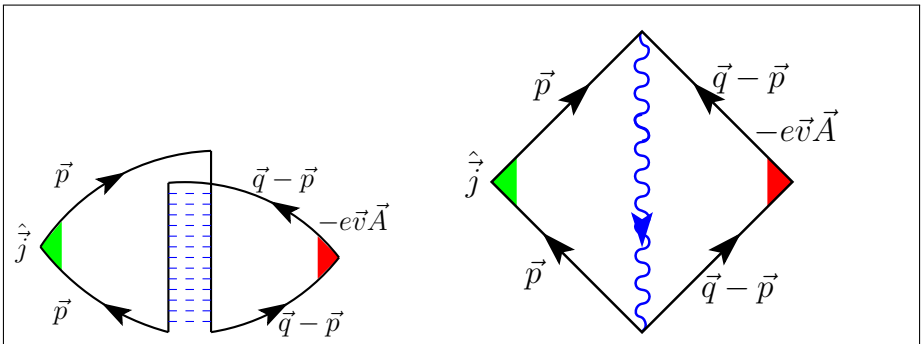
$$\forall E \quad \frac{1}{V}\sum_{\vec{p}}\frac{1}{(E-\varepsilon(\vec{p}))^2+\frac{1}{4\tau^2(E)}} = 2\pi\nu_E\tau_E \tag{3.33}$$

[12] Sometimes (6.8) $\hat{\vec{v}}\neq\hat{\vec{p}}/m$.

[13] Seems that some people mean other things [than (3.29)] by Lehmann representation, see p.[29]134.



(a) Main (Drude) contribution.



(b) Weak localization contribution. Two forms of the same diagram are shown, see Sec. 3.6.

Figure 3.1: Diagrams for calculating conductivity. The current and electric field vertexes conserve momentum. Because we have no diffuson or cooperon on Fig. 3.1(a), for $\omega \ll 1/\tau$ it is ω-independent. (ВК, 03.05.2002)

3.4 The weak localization correction

See the literature list in [4], and also (some of numerous) experiments: PRL**89**276803, [0703053]. In addition to the zeroth order diagram, considered in section 3, one can consider a sequence of "crossed" diagrams, each of the terms in which can be unbinded to a cooperon-like diagram [see fig. 3.1(b)]. This "cooperon" has a pole at $\vec{p}_1 = -\vec{p}_3$, so $\vec{p}_1$ and $\vec{p}_3$ are again correlated like in case considered in Sec. 3 .

In the weak localization diagram, we are lucky that only small values of q contribute to the result.[14] Imagine the opposite: in case of large q we could approximate cooperon line with it value on $q \sim 1/l$, and then perform 2 independent integrations: by $\vec{p}$ and by $\vec{q}-\vec{p}$. This would mean that the Hikami box[15] on the right of Fig. 3.1(b) splits into two bubbles. The largeness of an additional bubble is compensated by the smallness of the coefficient of the cooperon, but still we would have the diagram of the same order, as the main contribution to the conductivity! Fortunately, this is not the case: both of two bubbles have only one vector vertex, which makes them zero.

From (3.11) and (2.7) we get the correction to the conductivity for zero temperature technique

$$\delta\sigma = -\operatorname{sign}\omega \frac{e^2 D}{\pi} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{-i|\omega| + Dq^2},$$

We see here again strange sign - type singularities, like in section 3.1. As before, Matsubara technique is free from them:

$$\clubsuit \quad \delta\sigma = -2e^2 D \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{-i\omega + Dq^2} \tag{3.34}$$

One can see that (3.34) produces negative correction to the conductivity, which can be considered as decreasing propagating ability of electrons. This is a reason to call (3.34) the WL correction[16] To calculate the integral in (3.34) we should substitute[17] $-i\omega$ by $-i\omega + 1/\tau_\varphi$. Also we should introduce a cut-off, upper limit[18] for integrals by q: $l_0 \sim l$. Finally

[14] This is not completely true: in 2D, in the wide region of values of the integration variable q, the integrand behaves as $1/q$, so we get log. However, the upper cut-off $\sim 1/l$ is not important; see also below in this Sec.

[15] The physical sense of a HB is explained on p. 71, Fig. 14 of arXiv/0712.1154.

[16] Note that SOI may lead to positive corrections to the conductivity – "antilocalization" [34, 35].

[17] Note that the way of calculating τ_φ is still not clear. It occurs from evaluating the complete (that is **infinite**) perturbation series for the electron-electron interaction. Its physical sense: in mesoscopic systems, we always look for coherence effects. The coherence is broken by the temperature fluctuations and by the interaction between the electrons. With each of these factors we can associate some length, on which the coherence is broken and compare these lengths with the size of the sample L. If these lengths are both larger then L, they (and corresponding effects of coherence breaking) are not important.

[18] In reality we always have a cut-off at $q \sim 1/l$ our HBs behave like $\sim (1 + l^2 q^2)^{-1}$. We loose this cut-off when we assume that the diffusion approximation holds.

we get for $\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{-i\omega + Dq^2}$ (cf. [1] and p. [36]282-283)

$$\begin{aligned} &\text{in (quasi) 1D} && \frac{\sqrt{\tau_\varphi \tau}}{2l}, \\ &\text{in 2D} && \frac{\tau}{2\pi l^2} \ln\left(1 + \frac{1}{2}\frac{\tau_\varphi}{\tau}\frac{l^2}{l_0^2}\right), \\ &\text{in 3D} && \frac{3\tau}{2l^3\pi^2}\left\{\frac{l}{l_0} - \frac{1}{l}\sqrt{\frac{3\tau}{\tau_\varphi}}\arctan\left(\sqrt{\frac{\tau_\varphi}{3\tau}}\frac{l}{l_0}\right)\right\}. \end{aligned} \tag{3.35}$$

From this it is easy to see that

$$\frac{\delta\sigma}{\sigma} \sim \frac{\tau}{l^d \nu_0} \sim \frac{1}{D\nu_0 l^{d-2}}. \tag{3.36}$$

In the future, the following will one of our parameters of smallness[19]:

$$\frac{\delta\sigma}{\sigma} = \frac{1}{\nu_0}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{-i\omega + Dq^2}. \tag{3.37}$$

Our considerations are correct only if (3.36) is small. STOP BTW, where could one find the proof[20] of the fact that in 1D there is always localization? We usually work in 2D, which is special, since an infinite 2D sample is always localized, but finite ones – not neccesary [37].

A small resume: When considering the average of two Green functions, one should always keep in mind that actually there are always two contributions: cooperon-like one and the diffuson like. In our particular case the zeroth order gives Drude formula (3.14), diffuson contribution is equal to zero, and cooperon contribution gives weak localization (3.34).

3.5 How everything is simple in Keldysh technique

Eqs. (3.2), (3.3), (3.4), ([5]8.9) and (3.5) hold also for the Green function G_{C_K} and ordering T_c originally defined in Keldysh technique, see (9.1). One can show that (3.5) is also conserved by the "matrix" isomorphism (9.2).

Note that operators in quantum mechanics are introduced in the same manner, as in [38], when the fluctuations are discussed: we invent a hermitian operator, which has the average equal to the quantity in which we are interested in. Once these conditions are fulfilled, we say this is an operator of the considered quantity. In this way becomes absolutely clear the derivation of (3.4). From (3.5) we arrive to (3.8) note that here

$$\begin{aligned} &\delta^{(1)} G(\vec{p}; E, E-\omega) = G(\vec{p}, E)\left[-\frac{e}{c\hbar}\vec{v}\vec{A}_\omega\right] G(\vec{p}, E-\omega), \\ &\delta G_K(\vec{p}; E, E-\omega) = (h_E - h_{E-\omega})\, G_R(\vec{p}, E)\left[-\frac{e}{m\hbar}\vec{v}\vec{A}_\omega\right] G_A(\vec{p}, E-\omega), \\ &\text{so that} \quad \sigma = \nu D e^2. \end{aligned} \tag{3.38}$$

STOP insert $\hbar$ in the last eq.

The diamagnetic term $\propto \vec{A}$ in the current operator (3.2) is cancelled by

$$\delta(G_R - G_A)(\vec{p}; E, E-\omega) = [G_R(\vec{p}, E) G_R(\vec{p}, E-\omega) - G_A(\vec{p}, E) G_A(\vec{p}, E-\omega)]\left[-\frac{e}{m\hbar}\vec{v}\vec{A}_\omega\right]. \tag{3.39}$$

From (3.38) and (3.39) we see that δG_K *is not* proportional to $\delta(G_R - G_A)$. This contradicts a usual belief that in linear responce "everything is like in equilibrium".

Somewhat more tricky is the calculation of the correction of the conductivity due to the interaction. Now we have a perturbation operator consisting of two terms[21]: (a) external electric field and (b) the interaction. Because our measurement (current) operator is a vector one, only terms with odd numbers of our electric field vertices produce

[19] cf with $\mathbb{G}$ (3.22) and g (3.41).

[20] Найти и рассказать об этом Б. Н. Захарьеву.

[21] Somewhat similar situation had been already considered in Sec. 2.1

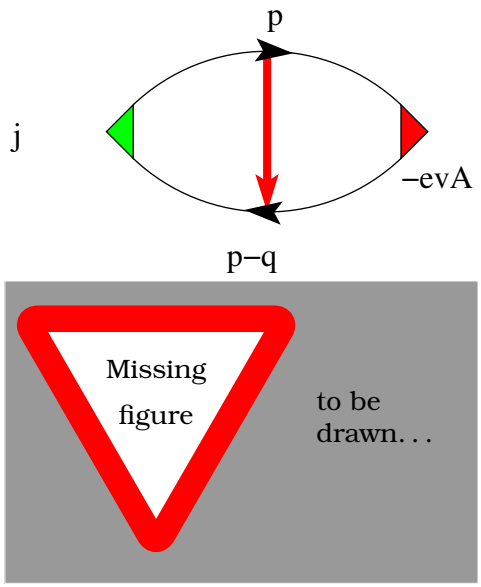


Figure 3.2: Correction to conductivity due to the interaction (see `condCorr.nb`).

non-zero results. So, the largest non-zero correction should be the one that comes from the mixed term in the second order of perturbation theory: $\left(\hat{V}_1(t_1)+\hat{V}_2(t_1)\right)\left(\hat{V}_1(t_2)+\hat{V}_2(t_2)\right) \to \hat{V}_1(t_1)\hat{V}_2(t_2)+\hat{V}_2(t_1)\hat{V}_1(t_2)$. We see that effectively we have commutating[22] operators: $\forall t_1,t_2\ \left[\hat{V}_1(t_1),\hat{V}_2(t_2)\right]_- = 0$, where $\hat{V}_1(t)$ is given by (3.4) and $\hat{V}_2(t)$ - by ([5]6.4) with all $\psi(\vec{r}) \to \psi(\vec{r},t)$ (the same for $\psi^\dagger$; t is one and the same for all ψ and $\psi^\dagger$).

One could naïvely think that it is possible to perturb with $\hat{V}_2$ Green functions in diagrams for $\delta G^{(1)}$ already perturbed with $\hat{V}_1$; however this would lead to the loss of the diagram from Fig. 3.2.

Literature to learn more on the topic:

- Scaling theory of localization PRL**42**673. Вроде бы это есть также и в книжке Гантмахера.
- Correction to the conductivity due to interaction: [39, 40].
- Hall effect: [41].
- Weak localisation correction $\Delta\rho_{xy}=0$: [42], as well as my own (easy) calculation.

3.6 The loop expansion

This Sec. is written in [20]; see also Sec.[43]III.3.c and especially p.[43]296. During my talk in Korea there were several questions about the loop expansion. Apparently it is not described in the known literature, so let me do it here.

There are two ways of representing diagrams in the disorder averaging technique. Consider two diagrams in Fig. 3.1(b). The first diagram is understood by a larger number of people than the second one. It's cooperon is represented as a ladder. This way has intuitive connection with the calculation of the cooperon; it reminds the fact that a cooperon is represented by an infinite series of diagrams, see Sec. 2. However, once we have learned how to calculate a cooperon, it is better to represent diagrams in an alternative way: by taking the inverse Fourier transformation of a Green function, it is easy to notice that, in coordinate space, our averaged Green functions are short-range objects: e.g., in 3D $G_{R/A}(\vec{r},\vec{r}\,') \propto \exp[|\vec{r}-\vec{r}\,'|/l]$. Thus, thinking now in coordinate space, we can say that the length of a Green function line is l. Since in the disorder averaging technique we are unable to observe any effects on scales shorter than l, to say that the length of a Green function line is l, is the same, as to say that this legth is zero, which means that we can consider it not as a line, but as a point.

Consider now the lhs-diagram in Fig. 2.2(a), which depicts a cooperon in its "usual form". Being an average of two Green functions, a cooperon has four external line ends. From Sec. 2.2 we know, that (in the coordinate space) the distance between points 1 and 2 is of the order of an impurity size (i.e., practically zero). This is the reason for us

[22] and this is the only qualitative difference from the ordinary case that I see.

to merge points 1 and 2 into one point; we have done it on the second (rhs) diagram in Fig. 2.2(a). Analogously, also points 3 and 4 in coordinate space get merged into one point. From Sec. 2.2 we know that the distance between 1 and 3 is much larger than l. Thus the second diagram in Fig. 2.2(a) reminds us about the spatial structure of a cooperon: in a coordinate space, a cooperon has only two external line ends. The explanation for the diffuson in Fig. 2.2(b) is analogous.

Based on these arguments, we give the recipe to transform any diagram from its "usual form" to its "coordinate form": just transform all CD lines into their coordinate form according to[23] Fig. 2.2. The simpliest example of one and the same diagram in two different forms is the WL diagram depicted on Fig. 3.1(b). More complicated example is the diagram in Fig. 11.5, who's "coordinate form" is the first diagram in Fig. 11.3(b).

Ok, now we can transform any diagram from its "usual form" into its "coordinate form". The "coordinate form" looks better from estetical point of view, but this is not its main advantage. Let me now finally tell what loops do I mean, when speaking about the loop expansion. If I take a diagram in its "coordinate form" and squeeze all Green function lines into points, the result will contain only CD lines, which will form a certain number of loops. For example, a bubble in Fig. 3.1(a) has obviously no loops (since it has no CD lines which could form a loop); a WL diagram in Fig. 3.1(b) (on the right) has one loop, and both diagrams on Fig. 11.3(b) have two loops.

Why the number of these loops is so important? Because (see Sec. 12)

$$\begin{aligned}(\text{number of loops}) =&(\text{number of CD lines}) - (\text{number of GFBs}) + 1 = \\ =&(\text{number of independent "small" momentum variables}).\end{aligned} \tag{3.40}$$

Here by "large" momentum variable I mean, e.g., an internal momentum $\vec{p}$ of a GFB; its characteristic value (given by the convergence scale of $\int \mathrm{d}^d p/(2\pi)^d$) is $p \lesssim 1/l$. Normally (except for the case of ZLA, when there are no CD-lines carrying non-zero momentum) we prefer to work in the diffusion approximation. (This permits us to assume that $ql \ll 1$ during the calculation of the CDs and GFBs). That is why we say, that the "mass" standing in the denominator of a CD, is small, so that $q^* l \ll 1$, where q^* is a characteristic momenta of a CD; it corresponds to the momentum of a CD line, typically denoted by $\vec{k}, \vec{q}$; its characteristic value depends on the particular problem. Possible candidates are: $1/L$ (like in Sec. 10.4), or L_ω from Sec. 10.4 with $\omega \equiv T$ or $\omega \equiv \tilde{T}$. In case of SOI, it can be x/l [where x is defined in (6.3)]. The WL-correction represents an important exception[24] from this rule: in fact, from (3.36) and (3.37) we realize that in this case $q^* = 1/l$, and $q^* l$ is not a small parameter, so that the diffusion approximation, strictly speaking, is not valid. [See also the note about the cut-off for $\int \mathrm{d}q$ before (3.35).]

Let us compare two diagrams for the same physical quantity (this means, that they have the same number of external vertices[25] of the same nature). An estimate for a GFB is $\propto \nu\tau^{h-1}$, where h is the number of Green function lines composing the GFB[26]. Every CD line $\propto (2\pi\nu\tau)^{-1}$. Let us estimate ν according to (13.11). Let us denote $L_{1,2}$, $H_{1,2}$, $C_{1,2}$ number of loops, GFBs, and CD-lines in the two considered diagrams; quantities h_{1j} denote number of GF lines in the jth GFB of the first diagram, and h_{2n} do the same in the second diagram.

$$\begin{aligned}\frac{\text{1st diagram}}{\text{2nd diagram}} \sim \frac{\int \prod_{i=1}^{L_1} \mathrm{d}^d k_i \left[\prod_{j=1}^{H_1} 2\pi\nu\tau^{h_{1j}-1}\right] \frac{1}{(2\pi\nu\tau)^{C_1}}}{\int \prod_{l=1}^{L_2} \mathrm{d}^d q_l \left[\prod_{n=1}^{H_2} 2\pi\nu\tau^{h_{2n}-1}\right] \frac{1}{(2\pi\nu\tau)^{C_2}}} = \boxed{L = C - H + 1,\ \textstyle\sum_{j=1}^{H_1} h_{1j} - \sum_{n=1}^{H_2} h_{2n} = 2(C_1 - C_2)} = \\ = (2\pi\nu)^{H_1 - H_2 + C_2 - C_1} \times \tau^{H_2 - H_1 + C_2 - C_1} \times \tau^{2(C_1 - C_2)} \times (q^*)^{d(L_1 - L_2)} = \left[\frac{4\pi^2}{\lambda\!\!\!^{-2}} \frac{l}{p_\mathrm{F}} (\lambda\!\!\!^{-} q^*)^d\right]^{L_1 - L_2} = \\ = \boxed{\lambda\!\!\!^{-} \sim \tfrac{\hbar}{p_\mathrm{F}}} = \left[4\pi^2 \left(\frac{q^*}{p_\mathrm{F}}\right)^d p_\mathrm{F} l\right]^{L_1 - L_2} = \boxed{\text{in 2D}} = \left[4\pi^2 \frac{(q^* l)^2}{p_\mathrm{F} l}\right]^{L_1 - L_2},\end{aligned} \tag{3.41}$$

where we used (3.40)13.11). Note: from (3.41) we see that DA ($q^* l \ll 1$) is not necessary for the validity of the loop expansion (however, it is usually used out of the ZLA in order to simplify calculations cf. Sec. 13.2) The final note: to generate diagrams, some people who don't know how (or don't want) to use computers, use their brains in

[23] См. также мою презентацию на русском языке.

[24] Formally – yes, it is an exception, however, we are mainly interested from the MF and temperature dependence of L_φ, which occurs on the lower limit of the integral. In other words, nothing would be changed, if in $\int \mathrm{d}q$ we would set the upper limit not to $1/l$, but, e.g., to $1/(10l)$, given that $10l << L_\varphi$, so in reallity also in the WL-case we can say that $q^* l \ll 1$, if it is so important to assume this.

[25] E.g., all diagrams for the conductivity have 2 external vertices: one for the applied electric field, and another one which measures the current.

[26] Sometimes, like in (13.35), GFB gains additional smallness of the order of $q^* l \ll 1$, where q^* is the characteristic momentum of a CD line (i.e., "small" momentum variable). However, one has to calculate a GFB in order to reveal how much of $q^* l$ it has. As I noted, the characteristic value of q^* is non-universal – it depends on the problem. It is clear that both $p_\mathrm{F} l$ and $(q^* l)$ can be varied. My `estimates.nb`, written for a metal say that $p_\mathrm{F} l \sim (q^* l)^{-1} \sim 100$. So one should be careful.

an unpleasant way: at first they have to draw all possible diagrams in the "usual form", then redraw them in the "coordinate form", calculate the number of loops, and prey that no important diagrams are forgotten. If this brings you headache, follow my way instead: use the set of the procedures which I have developed for this purpose, see Sec. 12.

3.7 General thoughts on the precision of the diagrammatic technique

STOP This section has to be corrected in accordance with [20]. For energies close to E_F, the GFs are very different from GFs of the disorder-free system; the latter are strongly modified due to the disorder. In fact, the averaged GF is the result of the summation of infinite perturbation series (SCBA). The situation is different for energies far away from the Fermi level: the first-order correction to the disorder-free GF is $\delta^{(1)}g^E_r = U[g^E_r]^2 \ll g^E_r$ so that the disorder-free GF is only slightly changed. This behavior can be modelled by introducing energy dependence in $1/\tau \to 1/\tau(\xi) + 1/\tau_{ee}(\xi)$, where τ_{ee} is the relaxation time for the electron-type excitations in the Fermi liquid theory. The elastic relaxation time $\tau(\xi) = \tau$ is constant for $|\xi| \lesssim 1/\tau$; it increases with increasing $|\xi| > 1/l$. The inelastic relaxation time τ_{ee} is infinite at the Fermi level; it decreases with increasing $|\xi|$. Based on these arguments, we estimate the relation between "universal" and "non-universal" integrals:

$$\frac{\int_{-E_F}^{\infty} d\xi g_r^2}{\int_{-\infty}^{\infty} d\xi g_r g_a} \sim \frac{\min[E_F^2\tau_{ee}^2\big|_{\xi=-E_F}, 1]}{E_F\tau} \ll 1. \tag{3.42}$$

STOP Check with DL: We argue that even in a dirty material (but with high concentration of electrons) with $E_F\tau \sim 1$ the relation (3.42) still must be small. It would be of the order of 1 if one neglects interaction effects, which diminish the effect of external fields on the deep states. One can think of an analogy with a wind creating waves on the surface of a sea. The correction (3.42) is analogous to the effect of the bottom on the surface waves. This effect is large only if the wave amplitude is comparable to the depth. When the wind (external perturbation) is weak, it creates small waves which can be calculated neglecting interaction effects. One would strongly *overestimate* the wave amplitude created by a strong wind, if he neglects interaction between the water molecules. The interaction together with the gravity, stabilizes the see surface: if the depth is large, the interaction effectively makes it infinite. Analogously, interaction between electrons stabilizes the Fermi sphere against strong external perturbations (e.g., strong disorder). It only seems in our interaction-free calculation that the disorder affects the electrons near the bottom of the energy spectrum. In reality, because of the interaction, the effect of disorder is much smaller on deep electrons; that is, E_F in (3.42) is much larger than the Fermi energy.

During our calculations of diagrams in DAT we use the following approximations:

1. Ignoring diagrams with crossed IAL in the diagrammatic expansion for $\langle G_R G_A\rangle$ (where every IAL connects G_R with G_A). This is not an approximation, since every such "crossing" is included into another diagram, where an IAL is substituted with the ladder. This is analogous to how 1.1(b) is included into the diagram [0305478]5.c.i.

2. In the SCBA (see Sec. 1.4): ignoring diagrams with crossed IAL, which would result in the ξ- and (in case of SOI) spin-dependent corrections to τ in the GF's denominator.

3. $\int_{-E_F}^{\infty} d\xi \approx \int_{-\infty}^{\infty} d\xi \quad \Longrightarrow \quad \int \frac{d^2p}{(2\pi)^2} G_R^n(\vec{p}) \approx 0,\ n \geq 2$. This means ignoring non-universal corrections, see p. 30.

4. (without SOI) Average DoS≈ const – must work fine in 2D. This is the same assumption, as квадратичность (?=quadracity) of the energy dispersion. In fact, at least in semiconductors, electron dispersion law is not quadratic, see, e.g., this Chinese paper.

5. STOP Requires discussion with guru. Ignoring IAL with more than two ends, See pp. [36]80-81. see p.[5]428. This is really strange, because, apparently, averaging must be Gaussian, and for such an averaging we know that $\langle U(\vec{r})U(\vec{r}')\rangle \propto \delta(\vec{r}-\vec{r}') \Longrightarrow \langle U(\vec{r}_1)U(\vec{r}_2)U(\vec{r}_3)U(\vec{r}_4)\rangle \propto \delta(\vec{r}_1-\vec{r}_2)\delta(\vec{r}_3-\vec{r}_4) + \delta(\vec{r}_1-\vec{r}_3)\delta(\vec{r}_2-\vec{r}_4) + \delta(\vec{r}_1-\vec{r}_4)\delta(\vec{r}_3-\vec{r}_2)$, so that no lines with more than two ends can occur.[27] An attempt to study a diagram with a disorder line having 4 ends in the model (1.11) results in divergent integrals $\left[\int d\xi G_R(\xi)\right]^3$, which I don't know how to interpret.

[27] However, people draw lines with 3 ends, see, e.g., arXiv/0902.2571.

6. (Better call it not "approximation", but "assumption") Segregation of length scales (required for the loop expansion; see Sec. 3.6): the charachteristic GFB momentum must be much larger than characteristic CD momentum: $p \sim p_F \gg 1/l \gg \max[1/L, x/l]$, where x determines "mass" of a CD-line [e.g., in case of SOI x is given by (6.3).] In the simpliest case, when CD-lines are massless, the required segregation is provided by the diffusion approximation: $l/L \ll 1$.

7. Consider now lonely IALs, which we sometimes insert into HBs (cf. Figs. 7.2(b), 7.2(c), 11.1, etc). Such IAL can be (on both ends) appended with an additional IAL, connecting two GF, to which the considered ("main") IAL is attached. Such a structure can be considered as a "renormalized" IAL, which will weakly ($\propto (p_F l)^{-1}$) depend on the momentum which flows through it. This effect is similar to what we would have if we connect by a IAL two GFs (of the same type) belonging to different HBs. Thus, in order to make the procedure for generating diagrams self-consistent, we have to impose a rule: an addition of an IAL should not lead to $\propto (p_F l)^{-1}$-smallness. STOP Can these corrections be *ever* important? E.g., if the first loop contribution to some quantity is zero, can these corrections have the same order of magnitude as the contribution of two-loop diagrams?

Many (if not all) of these approximations have relative correction $\sim (E_F\tau)^{-1}$. Does it make sense then to calculate first and second loop corrections if I've already used all these approximations to calculate diagrams of the zero loop? E.g., the 1st-loop contribution can be of the same order, as corrections to all aboveenumerated approximations (which I've neglected!). Thinking in this way, one concludes that only calculating the ZLA makes sense (because I will die before I'll take into account corrections to all these approximations).

Fortunately, the situation isn't so bad. Concerning the 3rd approximation: When Altshuler-Aronov studied WL contribution to the conductivity, they considered only the WL diagram 3.1(b), which should be of the same order, as corrections to the Drude diagram 3.1(a). However, the integrand under $\int d\xi$ is MF-independent in the Drude-diagram, so it was clear that, even if they would calculate $\int_{-E_F}^{\infty} d\xi$ precisely, they would not obtain MF-dependence.

Read this section until the end, but let me make the conclusion now: We are always allowed to use the 3rd approximation (and, I suspect, other approximations, too). This permits us to use computer symbolic calculation programs (like *Mathematica* or `Maxima`) to calculate analytically huge expressions for complicated diagrams. We can only calculate the main contribution to a physical quantity. E.g., I don't know, how one could calculate corrections $\sim (E_F\tau)^{-1}$ to the WL correction, or, e.g., a correction $\sim f(x_\alpha, x_\beta)(E_F\tau)^{-2}$ to the SOI-dependent charge conductivity. [since from (8.13) I know that the leading SOI-dependent contribution is $\sim e^2/h \times (E_F\tau)^{-1}$].

3.7.1 Universal and non-universal contributions

Thoughts form 03.07.2008: from the discussion with DM about the SCBA we conclude that τ is independent on SOI.

Older stuff: See my email to DL 17:27 20.01.2008. Let us give a definition: we say that "integral $J = \int_{-E_F}^{\infty} f(\xi)d\xi$ converges on (the scale of) $\delta\xi$", if $J_{\delta\xi} \equiv \int_{-\delta\xi}^{\delta\xi} f(\xi)d\xi$ has the same order of magnitude as J. If $|J_{\delta\xi}| \ll |J|$, this means, that the integral does not converge on $\delta\xi$, but on some other scale $\gg \delta\xi$. Obviously, this definitions wouldn't make any sence for oscillating integrals, but we don't deal with them. (All our integrands are rational functions – products of GFs.)

In Eq. (3.41) we estimated GFB with h GF lines as $2\pi\nu\tau^{h-1}$. This is based on the assumption that $\int d\xi$ converges on the scale $\delta\xi \sim 1/\tau$ so that $\int d\xi G_R^n G_A^m \sim \tau^{n+m-1}$. The simplest example of such a GFB is a bubble $\nu \int d\xi G_R(\xi)G_A(\xi) = 2\pi\nu\tau$. It is *universal*, because only values of $\xi = \varepsilon_{\vec{p}} - E_F$ out of close vicinity of zero (i.e., $\varepsilon_{\vec{p}} \in [E_F - 1/\tau, E_F + 1/\tau]$) contribute to the result. Only in this close vicinity of the Fermi level Landau Fermi liquid theory (see pp.[44]22, [45]76, [8]25,31, [46]345,347,350, [47]395-400) is valid. Some integrals, e.g., $\int d\xi G_R^2(\xi)$, we call *non-universal*. We don't like them and we want to forget about them because (i) they are smaller (*at least* by the factor of $(E_F\tau)^{-1} \ll 1$) than universal integrals from GFB having the same number of GF lines, and (ii) we cannot calculate them, because we don't know the integrand in the important region of integration, i.e., for $\varepsilon_{\vec{p}} \ll E_F - 1/\tau$, see Sec. 1.4. [In addition, we don't know the energy spectrum and the density of states far away from E_F.]

The universal corrections are produced by (i) both real and imaginary parts of the $\int d\xi G_R^n G_A^m$, (ii) $\int d\xi \,\mathrm{Sp}\, \Im G_{R/A}^n$, and (iii) $\int d\xi \,\mathrm{Sp}\left[G_R^n - G_A^n\right]$. All other contributions we agree to neglect due to good physical reasons. Due to the same reasons we always approximate $\int_{-E_F}^{\infty} d\xi \approx \int_{-\infty}^{\infty} d\xi$. STOP

But what is even more important, long ago, before we started to calculate anything with the diagrammatic technique, we have sacrificed our ability of going far away from E_F in order to be able to ignore effects of interaction (or consider

them perturbatively). Long ago we have sworn to great god Landau that we never calculate any integral involving the states far away from E_F, and we must not forget about that.

I can not exclude that sometimes, e.g., when the quantity of interest is given by two-loop diagrams, it can happen, that non-universal corrections could be of the same order of magnitude as universal ones. However, I have never experienced such situations. Anyway, universal contributions are crucially different from the non-universal ones. Non-universal ones are much more individual[28], so if we found a non-zero universal effect it is extremely unlikely that non-universal corrections could cancel it (though may be could sometimes change sign).

In conclusion, we must always ignore non-universal integrals, because

- due to the invalidity of the SCBA for the values of ξ far away from E_F, we don't know the integrand. Electrons deep under the Fermi level can not elastically scatter off impurities, because they don't have free slots with the same energy. I expect that these "deep" electrons behave very similar to the dilute electron gas in the absence of the disorder; one could express such behavior with a ξ-dependent $\tau \equiv \tau_\xi$, such that $\tau_\xi \approx \tau_0$ =const for $|\xi| \lesssim 1/\tau_0$ and $\tau_\xi \to \infty$ for $\xi \to -E_F$.
- we have no reasons to believe that the loop expansion works for non-universal corrections.
- far away from the Fermi level, the Landau theory may not work; then the effects of the interaction between electrons may not be neglected.
- See #3 on p. 29 and the introduction in [20].

3.7.2 Precision in problems with SOI

Example from my experience: calculation of $\int \frac{d^2p}{(2\pi)^2}\vec{p}G_R(p,E)G_A(p,E-\omega)$ in Sec. 7.1. [For the moment let us forget about corrections to the 2nd approximation, and believe that our expressions (6.23) are absolutely precise.] From Sec. 7.3 we know that it is o.k. to employ 3rd approximation [see private comμnication with MSH], which also gives relative corrections $\sim (E_F\tau)^{-1}$. What is so special about the 3rd approximation?

Theorem 1 *Suppose $B(\xi,x)$ [or $B(\vec{p},x)$] is an integrand for some GFB; ξ [or $\vec{p}$] is the integration variable, and x represents parameters (frequencies, SOI amplitudes, etc). Suppose $\int_{-\infty}^{\infty} d\xi B(\xi,x)$ converges on $1/\tau \ll E_F$ (see Sec. 3.7 about what this means). Then*

$$\sqsupset \int_{-\infty}^{\infty} d\xi B(\xi,x) \neq 0 \Longrightarrow \int_{-E_F}^{\infty} d\xi B(\xi,x) = \left[1+\frac{a(x)}{E_F\tau}\right]\int_{-\infty}^{\infty} d\xi B(\xi,x), \tag{3.43}$$

and $a(x)$ contains no large parameter $E_F\tau \gg 1$. IMHO any GFB integral must converge on $\xi \sim 1/\tau$; I think, this property can be considered as a part of the GFB's definition.

Note that because of the Sec. **??** I have doubts that (3.43) could be rewritten for $\int d^dp$. I'll prove (3.43), when time permits... Once we believe in (3.43), it is clear, why it is possible to use 3rd approximation for both diagrams in Fig. 7.1.

Now let us explore more interesting case of SOI-corrections to the charge conductivity in Sec. 8, see (8.7). This case is more complicated, since the corrections are small compared to the main (Drude) conductivity. The recipe: as ususally, calculate every GFB of a diagram separately, but two times: first perform $\int d\xi$ for the integrand with SOI amplitudes set to zero, obtaining the main contribution to this GFB. The result will not be precise because of the 3rd approximation involved; however, the corrections are going to be *x-independent*. Such corrections can not spoil our party.

Then, we integrate the integrand with the Drude (i.e., x-independent) contribution subtracted. When the main contribution $\int_{-\infty}^{\infty} d\xi B(\xi,x) \neq 0$, the situation is standart: we can employ 3rd approximation. A dangerous situation is when the main contribution is zero, and not due to the matrix structure of the GFB or due to the angle integration, but just because of the 3rd approximation. (note that *at first* I calculate matrix part of Sp together with its angle part, $\int d\Omega_{\vec{p}}$, and only then I take $\int d\xi$) Such a situation would require special treatment (until now I've never faced it).

Based on this, we realize that we can employ the 3rd approximation when calculating the diagram in Fig. 7.1(b). But what about more complicated ones, e.g., on rhs of Fig. [20]**??**? There GFB depends momenta $\vec{k}$ of CD lines – these momenta are part of x in (3.43). Thus, after we already calculated the GFB [i.e., integrated over its "main" big moment $\vec{p}$ ($|p-p_F| \sim 1/l$)], we will integrate it over $\vec{k}$. To estimate the precision of the final result, we will have to integrate $a(x) \equiv a(y,\vec{k})$

[28] BTW, we also have no grounds to believe that DAT works for non-universal contributions.

over $\vec{k}$. Will this integration make our small correction large? No, because $\vec{k}$ is a "small" momentum variable. In fact, the $\vec{k}$ of the GFB will be always expressed in terms of kl, so in the 6th assumption $kl \ll 1$ our small correction has no chance to become ever important. (unless $\int \mathrm{d}^d k$ kills the main contribution which should never happen)

3.7.3 About SCBA

STOP After the discussion with DM, this section has to be completely rewritten. (See Sec. 1.5 and 6.9.) Now its time to remmember that our expressions for GF are not absolutely precise, because of the 2nd approximation (SCBA) used. In fact, DL was attacking me saying (in my approximate interpretation with additions) the following: "Let us now calculate self energy for the self-consistent Born approximation in more honest way, without approximating $\int_{-E_F}^{\infty} \mathrm{d}\xi \approx \int_{-\infty}^{\infty} \mathrm{d}\xi$ and also taking corrections to the 2nd approximation into account. We will obtain spin-dependent τ (i.e., not a scalar, but a 2×2 matrix) The spin-dependent correction is expected to be relatively small as $1/(E_F\tau) \sim 1/(p_F l)$. However, since the main contribution to the charge conductivity is by $(p_F l)^2$ larger, than the resulting spin-dependent correction to the conductivity, the effect of this spin-dependent τ has to be taken into account."

First, in order to get rid of blaming my τ in that it has spin-dependent corrections, I will consider *all* spin-dependent terms (i.e., Rashba and Dresselhaus SOI) in the Hamiltonian as a perturbation[29]. [Before I used GF (6.23) with implantated Rashba SOI, while the Dresselhaus part of SOI was considered as a perturbation.] In this way, my zeroth-order GF is a (independent of SOI amplitudes) scalar.

It's τ is a constant, if I permit myself using the 2nd approximation; otherwise τ has corrections like the one in Fig. 1.1(b). Due to them, τ may gain E- and $|\vec{p}| \equiv p$-dependence (which is expressed in terms of ξ-dependence):

$$\tau = \tau_0 + \delta\tau(p,E) \equiv \tau_0\left[1 + \frac{1}{p_F l} g\left(\xi\tau_0, E\tau_0\right)\right], \quad g(0,0) = 0, \tag{3.44}$$

where we have omitted constant part of the correction. The E-dependence will lead to temperature corrections of the final result; however, according to my `estimates.nb`, τ can be of about 800K (in a metal), so this can be safely disregarded. As for the p-dependence, the analogous of (3.43) can be utilized: (reminder: *at first* I calculate matrix part of Sp together with its angle part, $\int \mathrm{d}\Omega_{\vec{p}}$, and only then I take $\int \mathrm{d}\xi$)

$$\int_{-\infty}^{\infty} \mathrm{d}\xi B(\xi, \tau \equiv \tau_0) \neq 0 \Longrightarrow \int_{-E_F}^{\infty} \mathrm{d}\xi B[\xi, \tau \equiv \tau(\xi)] = \left[1 + \frac{\delta\tau}{\tau_0}\right] \int_{-\infty}^{\infty} \mathrm{d}\xi B(\xi, \tau \equiv \tau_0), \quad \frac{\delta\tau}{\tau_0} \sim \frac{1}{E_F\tau}$$

Seems obvious, that also the 4th approximation can be justified analogously.

Finally, I've justified all approximations except for the 5th one, which I still STOP have to understand. . .

3.8 Conductivity from the kinetic equation

See also: 0812.0024.

Let us take the simpliest KE ([21]2.80) for the simpliest case of δ-potential of impurity [so that $v(\vec{p}-\vec{p}\,') = U_0$ =const]. Note the definition of the distribution function ([21]2.77). STOP I must understand, how ([21]2.80) coincides with (9.5). Let us make the first change of variables in the distribution function, introducing $\tilde{f}(\varepsilon, \vec{n}, \vec{r}, t)$ according to

$$f(\vec{p},\vec{r},t) \stackrel{\mathrm{df}}{=} \tilde{f}\left[\frac{p^2}{2m} + U(\vec{r}) - E_F, \frac{\vec{p}}{p}, \vec{r}, t\right], \quad \frac{\partial f}{\partial \vec{p}} = \left[\vec{v}\frac{\partial}{\partial\varepsilon} + \frac{1}{p_F}\frac{\partial}{\partial \vec{n}}\right]\tilde{f}, \quad \frac{\partial f}{\partial \vec{r}} = \left[\frac{\partial U}{\partial \vec{r}}\frac{\partial}{\partial\varepsilon} + \frac{\partial}{\partial \vec{r}}\right]\tilde{f}, \tag{3.45}$$

so that the KE for $\tilde{f}$ looks like

$$\left[\frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{r}} - \frac{\partial U}{\partial \vec{r}}\frac{1}{p_F}\frac{\partial}{\partial \vec{n}}\right]\tilde{f} = \mathrm{St}[\tilde{f}], \quad \vec{p} \approx p_F\vec{n}. \tag{3.46}$$

Since the integration in $\mathrm{St}[f]$ is performed only along the direction of $\vec{p}$, $\mathrm{St}[f]$ is not changed: $\mathrm{St}[f] = \mathrm{St}[\tilde{f}]$. The advantage of (3.46) with respect to ([21]2.80) is that (3.46) shows explicitly energy conservation: neither the left part of (3.46), nor $\mathrm{St}[\tilde{f}]$ contain any operators, acting on variable ε. So let me from now on write ε as an index. We are interested in

[29] BTW, this speeds up the calculation a lot! Unless you don't want to go into very high order.

the steady situation, so that $\tilde{f}(\varepsilon,\vec{n},\vec{r},t)\to\tilde{f}_\varepsilon(\vec{n},\vec{r})$. I know that only anisotropic (in $\vec{p}$) part of $\tilde{f}$ is going to contribute into current. I am interested in the simplest $\omega=0$ case, so my current is conserved: it can not depend on coordinate. Then I conclude that also momentum-anisotropic part of f is coordinate-independent. In case of homogeneous $\vec{\nabla}U$, (e.g., when the applied $\vec{E}$ =const), I apply $\vec{\nabla}_{\vec{r}}$ to ([21]2.80) and obtain that $\Delta_{\vec{r}}f=0$. Solving $\Delta_{\vec{r}}f=0$ with the apprpriate BC, we get $\vec{p}$-isotropic part of (3.47).

Playing the same trick with $\tilde{f}$ and its KE (3.46) leads to

$$\sum_i\left\{v_{\mathrm{F}}n_i\frac{\partial^2}{\partial r_j\partial r_i}-\frac{1}{p_{\mathrm{F}}}\left[\frac{\partial^2U}{\partial r_j\partial r_i}-\frac{\partial U}{\partial r_i}\frac{\partial U}{\partial r_j}\frac{\partial}{\partial\varepsilon}\right]\frac{\partial}{\partial n_i}\right\}\tilde{f}=0,$$

which in case of $U=-eEx$ simplifies to

$$\sum_i\left\{v_{\mathrm{F}}n_i\frac{\partial^2}{\partial r_j\partial r_i}+\frac{\delta_{ij}(eE)^2}{p_{\mathrm{F}}}\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial n_i}\right\}\tilde{f}=0.$$

Note that BC for $\tilde{f}$ are different from BC for f, see Fig. 3.3.

My life experience [which consists of §[8]3.2-3, (9.39), p.[48]347, and Journal Club] tells me that the following ansatz[30] is good for $\tilde{f}$:

$$\tilde{f}_\varepsilon^{(0)}=f_\varepsilon^L\left(1-\frac{x}{L}\right),\quad \tilde{f}_\varepsilon(\vec{n},\vec{r})=\tilde{f}_\varepsilon^{(0)}+n_xEf_\varepsilon^1. \tag{3.47}$$

That's all I can imagine if I don't want to include higher orders of $\vec{E}$ in the ansatz.[31] (I am interested in the linear response.) Substituting (3.47) into ([21]2.80), and neglecting terms, quadratic in E, we obtain $f_\varepsilon^1=v_{\mathrm{F}}\tau(f_\varepsilon^L-f_\varepsilon^R)/(EL)$, so that $n_xEf_\varepsilon^1=(f_\varepsilon^L-f_\varepsilon^R)n_xl/L$, which contains smallness l/L. To obtain the current, we have to integrate the resulting $f_\varepsilon(\vec{n},\vec{r})$ by $\vec{n}$ and ε. But at first, just to understand how (3.47) coincides with the DF from §[8]3.3 (which is homogeneous), let rewrite (3.47) for $f(\vec{p},\vec{r})$ using (3.45):

$$f(\vec{p},x)=f^L_{\varepsilon(x)}-\frac{f^L_{\varepsilon(x)}-f^L_{\varepsilon(x)-eV}}{L}\left(x-\frac{p_x}{p}l\right)\approx\boxed{T\gg eV}\approx f^L_{\varepsilon(x)}-eE\frac{\partial f^L_{\varepsilon(x)}}{\partial E}\left(x-\frac{p_x}{p}l\right),\quad \varepsilon(x)\equiv\frac{p^2}{2m}-E_{\mathrm{F}}-eEx.$$

When calculating conductivity. we only need anisotropic in p part of $f(\vec{p},x)$, which is $eE\frac{\partial f^L_{\varepsilon(x)}}{\partial E}\frac{p_x}{p}l$, which is the same as in ([8]3.15)

Obviously the current given by (3.47), obeys charge conservation: $\operatorname{div} j=0$. STOP Почему-то получается в 2 раза больше – непонятно. I was not careful up to a numerical coefficients, but it is clear that it leads to qualitatively correct value $\propto e^2D$ for the Drude conductivity.

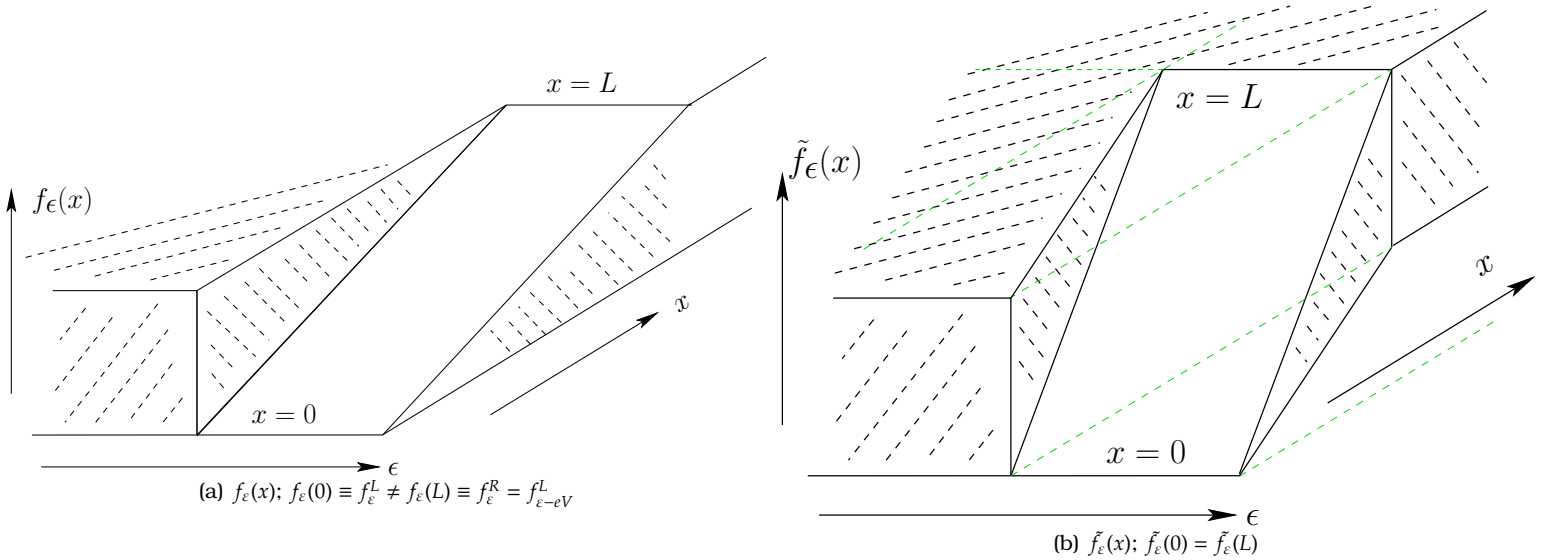
STOP What happens in the presence of SOI? See cond-mat/0510024 and this paper.

Can we use the usual form of the KE ([21]2.80) in order to obtain the same results, as from (more complicated than the "Drude bubble") diagrams with the number of loops ≥ 0? My answer: most probably, not. This is because the usual form of the KE ([21]2.80) corresponds to keeping only the largest term in the gradient expansion, which is used on p.[21]331-332 in the derivation of ([21]2.80). Gradient expansion is valid since we have "small" and "large" scales for momentum variables, see Sec. 3.6. In an infinite-size system with SOI, "small" momenta $q\sim x/l$; since $G_{\mathrm{R/A}}(\vec{r},\vec{r}\,')\sim\exp[ip_{\mathrm{F}}(\vec{r}-\vec{r}\,')+|\vec{r}-\vec{r}\,'|/l]$, there are two candidates for the role of "large" momentum scale: $p\sim p_{\mathrm{F}}$ or $p\sim 1/l$. That is, small parameter of the gradient expansion is $pq\sim x/(p_{\mathrm{F}}l)$ or $pq\sim x$. Thus, if, e.g., like in [49], we are interested in calculating corrections $\sim\sigma_Dx^2/(p_{\mathrm{F}}l)$, we can not use the usual form of the KE ([21]2.80) without extra justification. Without such a justification, we need to keep higher derivatives in the gradient expansion, which is going to change the form of the kinetic equation.

E.g., can we obtain the weak localization correction? –Probably not due to analogous reasoning.

[30] Note that the real life is reacher than (3.47), see p.[8]61.

[31] See §[8]3.2; BTW, $1/\tau=\int W(\vartheta)\,[1-\cos(\vartheta)]\,\mathrm{d}\Omega/\Omega_0$ ([8]3.11) is not true in 2D.



(a) $f_\varepsilon(x)$; $f_\varepsilon(0) \equiv f^L_\varepsilon \neq f_\varepsilon(L) \equiv f^R_\varepsilon = f^L_{\varepsilon - eV}$

(b) $\tilde{f}_\varepsilon(x)$; $\tilde{f}_\varepsilon(0) = \tilde{f}_\varepsilon(L)$

Figure 3.3: f_ε and $\tilde{f}_\varepsilon$. Thin green dashed lines are parallel to coordinate axes. Steps in energy dependence of $f_\varepsilon(x)$ result in the steps in coordinate dependence of $\tilde{f}_\varepsilon(x)$. The current of electrons flows in the direction from $x = L$ to $x = 0$.

Chapter 4

Non-linear σ-model

4.1 Keldysh technique in terms of the functional integrals

The main textbook is [50]. For the moment please assume everywhere that $\hbar \equiv 1$. Let us choose some moment t_0 in the past (following Kamenev, we will assume that $t_0 = -\infty$, remembering that this is *unnecessary*). The density matrix evolves according to[1]

$$\rho_t = U(t,t_0)\rho_{t_0}U^\dagger(t,t_0), \quad i\hbar\frac{\partial U(t,t_0)}{\partial t} = H_t U(t,t_0), \quad U(t_0,t_0) \equiv 1, \quad \forall t\ U(t+\delta_t,t) = \exp\left[-\frac{i}{\hbar}\hat{H}\delta_t\right],\ \delta_t \to 0.$$

The expectation value of some physical quantity O in the time moment t is

$$\bar{O} = \mathrm{Sp}\left[\hat{\rho}_t\hat{O}\right] = \mathrm{Sp}\left[\hat{\rho}_{t_0}\hat{O}_t\right] = T_C\left\{\mathrm{Sp}\left[\hat{U}_C\hat{\rho}_{t_0}\hat{O}\right]\right\}, \quad \hat{O}_t \equiv U^\dagger(t,t_0)\hat{O}U(t,t_0), \tag{4.1}$$

where $T_C\{\ldots\}$ is the closed-contour-ordering operator, see Fig. 9.1.

In the momentum-coordinate representation matrix element of the evolution operator is ([51]2.48)

$$\langle x_2|U(t_2,t_1)|x_1\rangle = \int D[x(t)p(t)]\exp\left[\frac{i}{\hbar}s(p,x)\right], \quad s(p,x) = \int_{t_1}^{t_2}[p\dot{x} - H(p,x)]\,\mathrm{d}t. \tag{4.2}$$

Analogously, in the space of Fermionic coherent states

$$\langle\varphi_2|U(t_2,t_1)|\varphi_1\rangle = \int D[\bar{\varphi}(t)\varphi(t)]\exp\left[\frac{i}{\hbar}s(\bar{\varphi},\varphi)\right], \quad s(\bar{\varphi},\varphi) = \int_{t_1}^{t_2}[\bar{\varphi}\dot{\varphi} - H(\bar{\varphi},\varphi)]\,\mathrm{d}t. \tag{4.3}$$

In the Keldysh technique $\varphi_2 \equiv \varphi_1$ and instead of $\int_{t_1}^{t_2}\mathrm{d}t$ we substitute $\oint\mathrm{d}t$. The trace of the evolution operator over the closed time contour 9.1 is equal to

$$\begin{aligned}\mathrm{Sp}\,\hat{U}_C = \int \mathrm{d}\bar{\varphi}_1\mathrm{d}\varphi_1\langle\varphi_1|U(t_1,t_1)|\varphi_1\rangle e^{-\bar{\varphi}_1\varphi_1} = \lim_{N\to\infty}\int \mathrm{d}\bar{\varphi}_1\mathrm{d}\varphi_1\ldots\int \mathrm{d}\bar{\varphi}_{2N}\mathrm{d}\varphi_{2N}\\ \langle\varphi_1|\ldots|\varphi_{2N}\rangle\langle\varphi_{2N}|\ldots|\varphi_{2N-1}\rangle\ldots\langle\varphi_{N+1}|\ldots|\varphi_N\rangle\langle\varphi_N|\ldots|\varphi_{N-1}\rangle\langle\varphi_2|\ldots|\varphi_1\rangle\ldots\prod_{i=1}^{2N}e^{-\bar{\varphi}_1\varphi_1}.\end{aligned} \tag{4.4}$$

Let us define discretization step δ_t according to

$$\forall i = 2\ldots N\ t_1 - t_{i-1} = \delta_t > 0, \quad \forall i = N+2\ldots 2N\ t_1 - t_{i-1} = -\delta_t < 0, \quad \text{for } i = 1, N+1\ t_1 - t_{i-1} \equiv 0\ ,\ t_0 \equiv t_{2N}.$$

Keeping in mind that

$$\left\langle\varphi_i\left|\exp\left[-\frac{i}{\hbar}\hat{H}\delta_t\right]\right|\varphi_{i-1}\right\rangle \approx \langle\varphi_i|\varphi_{i-1}\rangle\exp\left[-\frac{i\delta_t}{\hbar}H(\bar{\varphi}_i,\varphi_{i-1})\right] = \exp\left[\bar{\varphi}_i\varphi_{i-1}\right]\exp\left[-\frac{i\delta_t}{\hbar}H(\bar{\varphi}_i,\varphi_{i-1})\right],$$

[1] Later, when we start the NLσM-derivation (see Sec. **??**) we will assume that unperturbed GFs are homogeneous in time; for that ρ_{t_0} must commute with the unperturbed Hamiltonian. But until that point, no restrictions on the DM ρ_{t_0} are required.

we get discrete representation of $s(\bar\varphi,\varphi)$:

$$\frac{i}{\hbar}s(\bar\varphi,\varphi)=i\bar\varphi^T g^{-1}\varphi,\quad \bar\varphi^T=(\bar\varphi_1\dots\bar\varphi_N,\bar\varphi_{N+1}\dots\bar\varphi_{2N}),\quad \varphi^T=(\varphi_1\dots\varphi_N,\varphi_{N+1}\dots\varphi_{2N}),$$

where, e.g., for $N=4$
$$ig^{-1}=\left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 1-\frac{i\delta_t}{\hbar}H & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1-\frac{i\delta_t}{\hbar}H & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1-\frac{i\delta_t}{\hbar}H & -1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\frac{i\delta_t}{\hbar}H & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\frac{i\delta_t}{\hbar}H & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{i\delta_t}{\hbar}H & -1 \end{array}\right],$$

where $H\equiv\{H_{ij}\}$ is the matrix of the *unperturbed* Hamiltonian:

$$\forall 1\le i,j\le 2N\quad \bar\varphi_i H\varphi_j\equiv\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{p^2}{2m}\bar\varphi_i(\vec p)\varphi_j(\vec p)+\int\mathrm{d}^dr U_{\mathrm{dis}}(\vec r)\bar\varphi_i(\vec r)\varphi_j(\vec r). \tag{4.5}$$

It is nice to see that the rhs of (4.4)

$$\int D[\bar\varphi(t)\varphi(t)]\exp\left[i\bar\varphi^T g^{-1}\varphi\right]=\det\left(-ig^{-1}\right)=\lim_{N\to\infty}O(N\delta_t^2)=0$$

is the same as the lhs of (4.4):

$$\mathrm{Sp}\,\hat U_C=\int\mathrm{d}\bar\varphi_1\mathrm{d}\varphi_1\left\langle\varphi_1\left|U_c(t_1,t_1)\right|\varphi_1\right\rangle e^{-\bar\varphi_1\varphi_1}=\int\mathrm{d}\bar\varphi_1\mathrm{d}\varphi_1\exp[\bar\varphi_1\varphi_1]\exp[-\bar\varphi_1\varphi_1]=0.$$

In the equilibrium case

$$\bar\psi_i\hat\rho_0\psi_j=\bar\psi_i\frac{\exp\left[-\tilde H/T\right]}{\mathrm{Sp}\exp\left[-\tilde H/T\right]}\psi_j\equiv\int\frac{\mathrm{d}^dp}{(2\pi)^d}\int\frac{\mathrm{d}^dp'}{(2\pi)^d}\frac{\bar\psi_{i\vec p}\exp\left[-\tilde H/T\right]_{\vec p\vec p'}\psi_{j\vec p'}}{\mathrm{Sp}\exp\left[-\tilde H/T\right]},\quad \tilde H\equiv H-\mu\mathbb{1}. \tag{4.6}$$

Thus, when calculating expectation values like (4.1) we meet the evolution operator $\hat U_C$ always in company of the DM $\hat\rho_0$. So, if we want to rewrite (4.1) in terms of the FI, in the upper argumentation we should substitute $s\to S$, $g\to G$, where matrix iG^{-1} differs from ig^{-1} by substituting "one" in the upper right corner with the unperturbed[2] DM,[3] see. ([50]5.4).

We split the integration over the closed time contour $\oint\mathrm{d}t$ in two ordinary time integrals, defining $\left\{\varphi_f(t_i)\right\}\Big|_{i=1}^N$ and $\{\varphi_b(t_i)\}\Big|_{i=1}^N$:

$$\varphi_f(t_i)\overset{\mathrm{df}}{=}\varphi_i,\quad \varphi_b(t_i)\overset{\mathrm{df}}{=}\varphi_{2N+1-i}\Longrightarrow\frac{i}{\hbar}S[\bar\varphi,\varphi]=i\bar\varphi G^{-1}\varphi=\int_{-\infty}^{\infty}\left[\bar\varphi_b\dot\varphi_b+i\bar\varphi_b H\varphi_b-\bar\varphi_f\dot\varphi_f-i\bar\varphi_f H\varphi_f\right]. \tag{4.7}$$

If we now change the notations, taking into account that $\varphi=(\varphi_f,\varphi_b)$, then in the continuous limit G^{-1} is described by the following 2×2 matrix in the (f,b)-space:

$$G^{-1}=\left(i\frac{\partial}{\partial t}-H\right)\sigma_3\equiv\left[\left(i\frac{\partial}{\partial t}-\mu\right)-\tilde H\right]\sigma_3,\quad S[\bar\varphi,\varphi]=\hbar\bar\varphi G^{-1}\varphi. \tag{4.8}$$

Already from that (4.8) doesn't contain any information about the energy distribution of electrons, it is clear that continuous representation of the matrix G^{-1} can not satisfy us: during the transformation from the discrete description to the continuous one we lost something infinitesimal, but important. Consequently, we can not utilize (4.8) for calculating GF. On the other hand, discrete representation is also not suitable for the practical usage.[4] We see that

[2] More precisely, with the DM-value at "initial" time moment t_0 in the past.

[3] I don't understand, why, differently from the bosonic case ([50]2.8), in ([50]5.4) in front of the DM stands minus sign. However, it is not important for what follows.

[4] I've tried to invert, and then to transform according to (9.2) discrete matrix G^{-1} for $N=4$, and I could not see that the result is the upper triangular matrix. However I've noticed that the inverse matrix $\propto\left[1-\rho(1+\delta_t^2H^2/\hbar^2)^{N-1}\right]^{-1}$.

the we've got into a dead end (=Sackgasse) – we have to "dig the tunnel" from the opposite side, i.e., dancing from calculated before (without FI) GF (9.1) and (9.2).

New variable change ([50]5.11,5.12):

$$\psi = L\sigma_3\varphi, \quad \bar\psi = L\bar\varphi, \quad \varphi = \sigma_3 L^T\psi, \quad \bar\varphi = L^T\bar\psi, \quad \bar\varphi^T G^{-1}\varphi = \bar\psi\left(LG^{-1}\sigma_3 L^T\right)\psi,$$

where L is defined in ([52]15) or in (9.2). GFs, calculated with the help of FIs using (for the moment unknown) *correct* matrix $\hat G^{-1}$, must coincide with GFs, obtained in a usual way (i.e., without using FIs). In other words, the following equations must hold:

$$LG^{-1}\sigma_3 L^T = \begin{pmatrix} \hat G_{\rm R} & \hat G_{\rm K} \\ 0 & \hat G_{\rm A}\end{pmatrix}^{-1} = \begin{pmatrix} \hat G_{\rm R}^{-1} & \left[\hat G^{-1}\right]_{\rm K} \\ 0 & \hat G_{\rm A}^{-1}\end{pmatrix} \Longrightarrow \tag{4.9}$$

$$\Longrightarrow G^{-1} = \frac{1}{2}\begin{pmatrix} \hat G_{\rm R}^{-1} + \hat G_{\rm A}^{-1} + \left[\hat G^{-1}\right]_{\rm K} & \hat G_{\rm R}^{-1} - \hat G_{\rm A}^{-1} - \left[\hat G^{-1}\right]_{\rm K} \\ \hat G_{\rm A}^{-1} - \hat G_{\rm R}^{-1} - \left[\hat G^{-1}\right]_{\rm K} & -\hat G_{\rm R}^{-1} - \hat G_{\rm A}^{-1} + \left[\hat G^{-1}\right]_{\rm K}\end{pmatrix}, \quad \left[\hat G^{-1}\right]_{\rm K} = -\hat G_{\rm R}^{-1}\hat G_{\rm K}\hat G_{\rm A}^{-1}, \tag{4.10}$$

where, according to ([50]2.30), the quantity $\left[\hat G^{-1}\right]_{\rm K}$ is infinitesimal, and, consequently, invisible in the continuous limit (4.8). We know that in GF (4.9) a chemical potential μ appears in the same way, as it appeared before in (4.6).

The result of above manipulations[5] is the regularization of the continuous limit of the matrix $\hat G^{-1}$: now we realize that instead of (4.8) one should use its regularized form (4.10), and even better – more convenient and well-known form (4.9). See the discussion in the end of Sec. [50]5.2.

4.2 Averaging over the disorder

In Sec. 4.1 we concluded that in case of non-interacting electron gas

$$S[\bar\psi,\psi] = \hbar\bar\psi G^{-1}\psi, \tag{4.11}$$

where instead of G one should substitute (4.9). Let us now add the disorder potential to our action. In (4.7) [see also (4.5)] the disorder potential can be hidden in H, and this helps us understanding that the disorder potential $U_{\rm dis}$ couples to the action as $V_{\rm cl}$ in ([50]116,118), i.e., through the unity matrix $\hat\gamma^{\rm cl} \equiv \sigma_o$. So it seems obvious that ([50]120) should hold – we did not assume the diagonality of the matrix H in Sec. 4.1. So I don't understand attempts of proving this anyway obvious statement in Kamenev's papers. So, we are switching on the disorder and we are going to average over it, like in Sec. [50]6.1. Instead of (4.11) one should use effective action[6]

$$\begin{aligned}\exp\left\{\frac{i}{\hbar}S\left[\bar\psi,\psi\right]\right\} &= \exp\left[i\bar\psi G^{-1}\psi\right]\int D[U_{\rm dis}]\exp\left\{\int {\rm d}^d r\left[-\pi\nu\tau U_{\rm dis}^2(\vec r) + iU_{\rm dis}(\vec r)\bar\psi(\vec r)\hat\gamma^{\rm cl}\psi(\vec r)\right]\right\} = \\ &= \exp\left\{i\bar\psi G^{-1}\psi - \frac{(\bar\psi\psi)^2}{4\pi\nu\tau}\right\}, \quad (\bar\psi\psi)^2 \equiv \int {\rm d}^d r\sum_{a,b=1}^{2}\int {\rm d}t{\rm d}t'\bar\psi_t^a(\vec r)\psi_t^a(\vec r)\bar\psi_{t'}^b(\vec r)\psi_{t'}^b(\vec r).\end{aligned} \tag{4.12}$$

As we know from the diagrammatics, dashed lines (representing averaging over the disorder) are quite similar to the lines of the electron-electron interaction with the difference that they don't transfer energy (frequency). We see this similarity in the action (4.12): it looks similar to the action of the interacting electron gas. Consequently, we treat it in the same way, as we would in case of the interaction – performing Stratonovich-Hubbard transformation[7] (introducing *matrix* field Q) and integrating over the Grassmann variables. This is all what we going to do until the end of this Section.

If we assume that all elements of the matrix Q are independent I have no idea how I could calculate the rhs of ([50]155). The situation improves if we assume that Q is a hermitian matrix [see between ([52]11b) and ([52]12)].

[5] Still this procedure hasn't completely satisfied me: seems that the difference of off-diagonal blocks of the discrete matrix ([50]5.4) should contain DM, but we don't see this in (4.10).

[6] Since $\bar\psi G^{-1}\psi$ and $\bar\psi\hat\gamma^{\rm cl}\psi$ are not matrices, but numbers (more precisely – products of even number of Grassmann variables which doesn't change the point), $\exp\left[i\bar\psi G^{-1}\psi + U_{\rm dis}\bar\psi\hat\gamma^{\rm cl}\psi\right] = \exp\left[i\bar\psi G^{-1}\psi\right]\times\exp\left[U_{\rm dis}\bar\psi\hat\gamma^{\rm cl}\psi\right]$.

[7] Stratonovich invented it two years before Hubbard.

Introducing notations:[8]

$$Q^{ab}_{tt'} = \bar{Q}^{ba}_{t't} \Longrightarrow \mathrm{Sp}\, Q^2 \equiv \sum_{a,b=1}^{2} \int \mathrm{d}^d r \int \mathrm{d}t\mathrm{d}t' Q^{ab}_{tt'}(\vec{r}) Q^{ba}_{t't}(\vec{r}) = 2 \sum_{a,b=1}^{2} \int \mathrm{d}^d r \int_{t\geq t'} \mathrm{d}t\mathrm{d}t' Q^{ab}_{tt'}(\vec{r}) \bar{Q}^{ab}_{tt'}(\vec{r}).$$

In this expression Sp is the product of three trace operators over three pairs of indices: Keldysh (a,b), time (t,t') and coordinate $\vec{r}$; trace over coordinates is just a simple integral, since $\forall a,b,t,t'$ the quantity $Q^{ab}_{tt'}$ is diagonal in coordinate space. In order to handle Q in the same way as G^{-1}, we introduce notation

$$\langle \vec{r},t,a|Q|\vec{r}',t',b\rangle \equiv Q^{ab}_{tt'}(\vec{r})\delta(\vec{r}-\vec{r}'), \quad \mathrm{Sp} \equiv \underset{\text{Keldysh time coordinate}}{\mathrm{Sp}\quad \mathrm{Sp}\quad \mathrm{Sp}}.$$

Then

$$\bar{\psi}Q\psi \equiv \sum_{a,b=1}^{2} \int \mathrm{d}t\mathrm{d}t' \bar{\psi}^a_t(\vec{r}) Q^{ab}_{tt'}(\vec{r}) \psi^b_{t'}(\vec{r}) = \sum_{a,b=1}^{2} \int_{t\geq t'} \mathrm{d}t\mathrm{d}t' \left\{ Q^{ab}_{tt'}(\vec{r}) \bar{\psi}^a_t(\vec{r}) \psi^b_{t'}(\vec{r}) + \bar{Q}^{ab}_{tt'}(\vec{r}) \overline{\left[\bar{\psi}^a_t(\vec{r}) \psi^b_{t'}(\vec{r})\right]} \right\},$$

and the integrand must not contain any singularities of the form $\delta(t-t')$, so that $t=t'$ is a set with zero measure in our two-dimensional time integrals. Let the indices α,β run all the values of the discretized set $\{t \leq t'\}$. Using[9] Tab. [51]1.1 (See p. [51]37) for $H_{\alpha\beta} = \frac{\pi\nu}{2\tau}\delta_{\alpha,\beta'}$ and $\eta = \frac{i}{2\tau}\bar{\psi}_t\psi_{t'}$, we get

$$\int D[Q] \exp\left[-\frac{\pi\nu}{4\tau}\mathrm{Sp}\, Q^2 + \frac{\bar{\psi}Q\psi}{2\tau}\right] = \exp\left[\frac{1}{2\pi\nu\tau}\sum_{a,b=1}^{2}\int \mathrm{d}^d r \int_{t\geq t'} \mathrm{d}t\mathrm{d}t' \left(\bar{\psi}^b_{t'}\psi^a_t\right)\left(\bar{\psi}^a_t\psi^b_{t'}\right)\right] =$$

$$= \exp\left[-\frac{1}{4\pi\nu\tau}\sum_{a,b=1}^{2}\int \mathrm{d}^d r \int \mathrm{d}t\mathrm{d}t' \left(\bar{\psi}^a_t\psi^a_t\right)\left(\bar{\psi}^b_{t'}\psi^b_{t'}\right)\right] \equiv \exp\left[-\frac{(\bar{\psi}\psi)^2}{4\pi\nu\tau}\right].$$

This new effective action[10]

$$\exp\left\{\frac{i}{\hbar}S[Q]\right\} = \int D[Q] \exp\left[-\frac{\pi\nu}{4\tau}\mathrm{Sp}\, Q^2\right] \int D[\bar{\psi}\psi] \exp\left[i\bar{\psi}\left(G^{-1} \pm \frac{iQ}{2\tau}\right)\psi\right].$$

We remember that nothing changes if we add a constant to the action and that $\log\det = \mathrm{Sp}\log$ and taking in the last expression the same sign as Kamenev has, we obtain ([50]157)

$$\frac{i}{\hbar}S[Q] = -\frac{\pi\nu}{4\tau}\mathrm{Sp}\, Q^2 + \mathrm{Sp}\log\left[G_0^{-1} + \frac{iQ}{2\tau} + V\right], \quad V \equiv \sigma_0 V_{cl} + \sigma_1 V_q, \tag{4.13}$$

where we for generality wrote some external (e.g., AC electric) field V.

4.3 Gapless excitations above the ground state

Because of the Sp the first variation can be easy calculated (like derivative from a function):

$$\forall n \in \mathbb{N} \quad \delta\,\mathrm{Sp}\, Q^n = n\,\mathrm{Sp}\left[Q^{n-1}\delta Q\right].$$

[8] We enlarged the set $\mathbb{C}$, introducing Grassmann variables in it. The bar over a symbol denotes complex conjugate if the symbol is a usual complex number, and (unusual) Grassmann conjugate if the symbol is a Grassmann variable. This construction is so durchsichtig that I believe that ([50]187) is valid also in case when J – is an element of our enlarged set. We follow the standard definition of Grassmann conjugate – see p. [53]61 and ([51]1.142b). Lerner in ([17]15) writes smth strange. In general, the title of [17] has affected its style – after reading [17] I felt being that "pedestrian" (BC explained that this is a politically correct synonym for "complete idiot").

[9] See also ([50]187).

[10] Don't forget that $\psi \equiv \psi(\vec{r},t)$ and $Q \propto \delta(\vec{r}-\vec{r}')$, but G^{-1} is not diagonal in coordinate space.

Let us calculate variation of the action in the vicinity of the stationary solution $\underline{Q}$ (where $\underline{Q}$ must be found from the condition $\delta S[\underline{Q}]=0$). For shortness we will omit the summation over Keldysh indices and over coordinates:[11]

$$\int \mathrm{d}t\mathrm{d}t' \left[\left(\underline{\bar{Q}}+\delta\bar{Q}\right)_{tt'}\left(\underline{Q}+\delta Q\right)_{tt'} - \underline{\bar{Q}}_{tt'}\underline{Q}_{tt'}\right] = \int \mathrm{d}t\mathrm{d}t' \left[\underline{\bar{Q}}_{tt'}\delta Q_{tt'} + \delta\bar{Q}_{tt'}\underline{Q}_{tt'}\right] = 2\int \mathrm{d}t\mathrm{d}t' \delta\bar{Q}_{tt'}\underline{Q}_{tt'},$$

$$\mathrm{Sp}\log\left[G_0^{-1}+\frac{i(\underline{Q}+\delta Q)}{2\tau}+V\right] - \mathrm{Sp}\log\left[G_0^{-1}+\frac{i\underline{Q}}{2\tau}+V\right] \approx \int \mathrm{d}t\mathrm{d}t' \left\{\left[G_0^{-1}+\frac{i\underline{Q}}{2\tau}+V\right]^{-1}\right\}_{tt'} \frac{i\delta\bar{Q}_{tt'}}{2\tau},$$

$$\Longrightarrow \frac{\delta\,\mathrm{Sp}\,Q^2}{\delta\bar{Q}_{tt'}} = 2\underline{Q}_{tt'}, \quad \frac{\delta\,\mathrm{Sp}\log\left[G_0^{-1}+\frac{i\underline{Q}}{2\tau}+V\right]}{\delta\bar{Q}_{tt'}} = \frac{i}{2\tau}\left\{\left[G_0^{-1}+\frac{i\underline{\bar{Q}}}{2\tau}+V\right]^{-1}\right\}_{tt'},$$

from where we understand why ([50]158) holds.[12] Please note: like in Sec. 1.5, also here arises the question about real part of the self energy, which is silently assumed to be zero in ([50]160).

If our action would be a finite-degree polynomial like on p. [54]81-86, then we would calculate also $\delta^2 S[Q]$, and then say that

$$S[\underline{Q}+\delta Q]-S[\underline{Q}] \approx \mathrm{Sp}\left\{\delta Q\circ\frac{\delta^2 S[\underline{Q}]}{\delta Q^2}\circ\delta Q\right\} = S_0+S_m,$$

where S_0 would correspond to Goldstone excitations (particles with zero mass). But, unfortunately, our action doesn't allow us to calculate $\delta^2 S[\underline{Q}]$, so that we have to approach the problem in an indirect way – through the Goldstone theorem. We hope[13] that our extremum

$$\underline{Q}_\omega \equiv \Lambda_\omega = \begin{pmatrix}1 & 2\mathcal{F}_\omega\\ 0 & -1\end{pmatrix} \quad \forall \mathcal{T}_\omega \in \left\{\left.\begin{pmatrix} r_\omega & \mathcal{F}_\omega(r_\omega-a_\omega)\\ 0 & a_\omega\end{pmatrix}\right| r_\omega, a_\omega \in \mathbb{C}\right\} \quad \mathcal{T}_\omega\Lambda_\omega\mathcal{T}_\omega^{-1}=\Lambda_\omega \tag{4.14}$$

corresponds to the minimum of the action (we call S "action" which is not quite rigorous). Note that *in the absence of perturbation* (i.e., for $V=0$) our action obeys continuous symmetry:[14]

$$\left\{Q\left|\, S[Q]=S\left[\underline{Q}_\omega\right]\right.\right\} = \left\{Q=\mathcal{T}\circ\Lambda\circ\mathcal{T}^{-1}\,\middle|\,\mathcal{T}\in\mathbb{T}\right\} \overset{\mathrm{df}}{=} Z, \quad \mathbb{T}=\left\{\mathcal{T}^{-1}=\mathcal{T}^\dagger\,\middle|\,\mathcal{T}_{tt'}(\vec{r})=\mathcal{T}_{t-t'}\right\}, \tag{4.15}$$

because matrices from the set $\mathbb{T}$ commute[15] with G_0^{-1} (but doesn't commute with Λ, unless they are proportional σ_0 in Keldysh space). Note that sets $\mathbb{T}$ and Z are isomorphic:

$$\mathcal{T}_1,\mathcal{T}_2\in\mathbb{T}, \quad \mathcal{T}_1\neq\mathcal{T}_2 \quad\Longleftrightarrow\quad Q_1,Q_2\in Z, \quad Q_1=\mathcal{T}_1\circ\Lambda\circ\mathcal{T}_1^{-1}\neq Q_2=\mathcal{T}_2\circ\Lambda\circ\mathcal{T}_2^{-1}. \tag{4.16}$$

I interpret ([50]6.10-6.11) as

$$\text{wrong statement: } Z=Z_N\overset{\mathrm{df}}{=}\left\{Q\,\middle|\,Q^2=1,\mathrm{Sp}\,Q=0\right\}. \tag{4.17}$$

In reality the set Z is smaller than Z_N.[16] Following the text after ([50]6.10), we call the set Z in (4.17) "Goldstone manifold". Analogously with the argumentation on p. [55]250-251 (i.e. fro the Goldstone theorem) we expect that Z will generate gapless excitations. More precisely, in our case the Goldstone theorem claims that in the expansion

$$S[Q]-S[\Lambda]\approx\mathrm{Sp}\,[Q\circ\Gamma\circ Q]\equiv$$

$$\equiv\sum_{a,b,c=1}^{2}\int\frac{\mathrm{d}^dp_1\mathrm{d}^dp_2\mathrm{d}^dq}{(2\pi)^{3d}}\int_{-\infty}^{\infty}\frac{\mathrm{d}E_1\mathrm{d}E_2\mathrm{d}\omega}{(2\pi)^3}\left\langle\vec{p}_1-\vec{q},E_1-\omega,a\left|Q\right|\vec{p}_1,E_1,b\right\rangle\left\langle\vec{p}_1,E_1,b\left|\Gamma_{\vec{q},\omega}\right|\vec{p}_2,E_2,c\right\rangle\left\langle\vec{p}_1,E_1,c\left|Q\right|\vec{p}_1-\vec{q},E_1-\omega,a\right\rangle, \tag{4.18}$$

$$\lim_{\vec{q},\omega\to0}\Gamma_{\vec{q},\omega}=0, \quad Q\in Z_N\overset{\mathrm{df}}{=}\left\{Q_\omega=\mathcal{T}\circ\Lambda\circ\mathcal{T}^{-1}\,\middle|\,\mathcal{T}\in\mathbb{T}_N\right\}, \quad \mathbb{T}_N=\left\{\mathcal{T}^{-1}=\mathcal{T}^\dagger\,\middle|\,\forall\mathcal{T}_{tt'}(\vec{r})\right\}, \quad Q_\omega^2=\mathbb{1}$$

[11] For the moment I don't see any small parameter which validate our stationary phase approximation. Or this is just a quasiclassical approximation? But then I placed $\hbar$ incorrectly.

[12] Spatial coordinates in the rhs of ([50]158) coincide because $Q\propto\delta(\vec{r}-\vec{r}')$. In our notations $\delta\bar{Q}_{tt'}$ is a coefficient in front of this δ-function.

[13] These hopes have grounds. Later we see that this is minimum (or at least – not maximum) on the subset $\mathbb{T}_N$; see. (4.18) and ([50]6.15). So, now we know that it is extremum and later we will see that it is not maximum. Thus, it is minimum.

[14] Like $\forall\mathcal{T}\in\mathbb{T}$, matrix ([50]6.9) is also diagonal in the frequency space (because we are interested only in stationary energy distributions), but not diagonal in Keldysh space – it has off-diagonal $2\mathcal{F}_\omega$, see. (4.14) and ([50]6.9). That's why for fixed Λ but different $\mathcal{T}$ in the rhs of ([50]6.10) we get different matrices Q.

[15] This claim is checked for Pauli matrices assuming that $\left[G_0^{-1}\right]_{\mathrm{R}}=\left[G_0^{-1}\right]_{\mathrm{A}}$ and $\left[G_0^{-1}\right]_{\mathrm{K}}$=0: then $\forall i=0\ldots3$ $[\sigma_i,G_0^{-1}]=0$.

[16] In fact, $Z_N\ni\mathcal{T}\circ\Lambda\circ\mathcal{T}^{-1}$ also for $\forall\mathcal{T}\notin\mathbb{T}$.

$\Gamma_{\vec{q},\omega}$ *approaches zero when* $\vec{q},\omega$ *both approach zero.* This statement is not obvious; in case of the vector field it is proved (no, not proved, rather "shown") on p. [54]81-86.[17] The set $\mathbb{T}_N$ is larger than $\mathbb{T}$: its elements $\mathcal{T}$ weakly depend on coordinate and on $t+t'$.[18]

We substitute in (4.13) ansatz for Q (4.18), multiply the log-argument by (Q-independent) constant G_0. Now we throw out of the (4.18) all terms, which are independent on Q. We get the new action[19]

$$\begin{aligned}\frac{i}{\hbar}S[Q] = \mathrm{Sp}\log\left[G_0^{-1}+\frac{i\mathcal{T}\Lambda\mathcal{T}^{-1}}{2\tau}+V\right] = \mathrm{Sp}\log\left[\mathcal{T}^{-1}G_0^{-1}\mathcal{T}+\frac{i\Lambda}{2\tau}+\mathcal{T}^{-1}V\mathcal{T}\right]\equiv\\ \equiv \mathrm{Sp}\log\left\{\left[G_0^{-1}+\frac{i\Lambda}{2\tau}\right]+\mathcal{T}^{-1}\left[G_0^{-1},\mathcal{T}\right]_- +\mathcal{T}^{-1}V\mathcal{T}\right\} = \mathrm{Sp}\log\{\mathbb{1}+A+B\}\approx \mathrm{Sp}\left\{A+B-\frac{A^2+B^2}{2}-AB\right\},\end{aligned} \tag{4.19}$$

where $A \overset{\mathrm{df}}{=} G\mathcal{T}^{-1}\left[G_0^{-1},\mathcal{T}\right]_-$ and $B\overset{\mathrm{df}}{=}G\mathcal{T}^{-1}V\mathcal{T}$. We perform calculations in the frequency space.

4.4 Isotropic diffusive metal

Calculating the trace in the momentum space, we use the following MEs of GF and of $\mathcal{T}$ and $\mathcal{T}^{-1}$:

$$\begin{aligned}\langle\vec{p}\omega|G_0^{-1}|\omega'\vec{p}'\rangle = \Big(\omega-\xi_p\Big)\delta(\omega-\omega')\delta(\vec{p}-\vec{p}'),\quad \langle\vec{r}\omega|\mathcal{T}|\omega'\vec{r}'\rangle = \mathcal{T}_{\omega\omega'}(\vec{r})\delta(\vec{r}-\vec{r}')\Longrightarrow\langle\vec{p}\omega|\mathcal{T}|\omega'\vec{p}'\rangle = \mathcal{T}_{\omega\omega'}(\vec{p}-\vec{p}'),\\ \langle\vec{p},E|Q|\vec{p}-\vec{q},E'\rangle = \int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{\mathrm{d}E_1}{2\pi}\langle\vec{p},E|\mathcal{T}|\vec{p}-\vec{k},E_1\rangle\Lambda_{E_1}\langle\vec{p}-\vec{k},E_1|\mathcal{T}^{-1}|\vec{p}-\vec{q},E'\rangle =\\ = \int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{\mathrm{d}E_1}{2\pi}\mathcal{T}_{E,E_1}(\vec{k})\Lambda_{E_1}\mathcal{T}_{E_1,E'}(\vec{q}-\vec{k}).\end{aligned} \tag{4.20}$$

We assume that in (4.19) one is much larger then the other terms in the log-argument, and we expand the log. In Keldysh space $G_0^{-1}\propto$ unity matrix: $G_0^{-1}=[\omega-\xi]$. The first order of expansion gives us

$$\mathrm{Sp}\left\{G\mathcal{T}^{-1}\left[G_0^{-1},\mathcal{T}\right]_-\right\} = \mathrm{Sp}\left\{G\mathcal{T}^{-1}\left[\omega,\mathcal{T}\right]_-\right\} - \mathrm{Sp}\left\{G\mathcal{T}^{-1}\left[\hat{\xi},\mathcal{T}\right]_-\right\},$$

If we calculate the first term in the coordinate space, then, due to (4.20), G under Sp will have coinciding arguments. Keeping in mind that $\forall\omega\ G^{\omega}(\vec{r}=\vec{r}')=-i\pi\nu\Lambda$, and that $\forall\omega\ \mathrm{Sp}\,[\omega\Lambda]=0$, we get[20]

$$\mathrm{Sp}\left\{G\mathcal{T}^{-1}\left[\omega,\mathcal{T}\right]_-\right\} = -i\pi\nu\,\mathrm{Sp}\left[\Lambda\mathcal{T}^{-1}\omega\mathcal{T}\right] = -i\pi\nu\,\mathrm{Sp}\left[\omega Q\right]\equiv\pi\nu\,\mathrm{Sp}\left[\partial_t Q\right]. \tag{4.21}$$

Note that the following matrix element depend only on the direction of the "main momentum" $\vec{n}$, but is independent on its modulus p:

$$\langle\vec{p}|[\xi,\mathcal{T}]|\vec{p}-\vec{q}\rangle = \mathcal{T}_{\vec{q}}(\xi_p-\xi_{\vec{p}-\vec{q}})\approx\mathcal{T}_{\vec{q}}v_{\mathrm{F}}\vec{n}\vec{q},\quad \int\frac{\mathrm{d}^dp}{(2\pi)^d}=\nu\int_{-\infty}^{\infty}\mathrm{d}\xi_p\int\frac{\mathrm{d}\Omega_{\vec{n}}}{\Omega_0}. \tag{4.22}$$

In other words, the differential operator $\hat{\xi}=p^2/(2m)-\mu=-\Delta/(2m)-\mu$ acts on the product of fast function $G(\vec{r})$ by slow functions $\mathcal{T}(\vec{r})$ and $\mathcal{T}^{-1}(\vec{r})$. From (4.22) it follows, that the result of action of $\hat{\xi}$ on $G(\vec{r})$ in our case is cancelled so that $\tau\hat{\xi}\sim q^2l^2\ll p_{\mathrm{F}}^2l^2$. Keeping in mind (4.22), we conclude that the first-order term in the expansion of the action (4.19) in ξ is zero:[21]

$$\mathrm{Sp}\left\{G\mathcal{T}^{-1}\left[\xi,\mathcal{T}\right]_-\right\} = v_{\mathrm{F}}\int\frac{\mathrm{d}^dp\mathrm{d}^dq}{(2\pi)^{2d}}G_p\mathcal{T}_{\vec{q}}^{-1}\mathcal{T}_{\vec{q}}\vec{n}\vec{q} = v_{\mathrm{F}}\nu\int\frac{\mathrm{d}\Omega_{\vec{n}}}{\Omega_0}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\vec{n}\vec{q}\int_{-\infty}^{\infty}\mathrm{d}\xi_pG(\xi_p)\mathcal{T}_{\vec{q}}^{-1}\mathcal{T}_{\vec{q}}=0, \tag{4.23}$$

so that the action (4.19) must be expanded up to the quadratic terms in ξ:

$$\mathrm{Sp}\left\{\left(G\mathcal{T}^{-1}\left[\xi,\mathcal{T}\right]_-\right)^2\right\} = \nu v_{\mathrm{F}}^2\int_{-\infty}^{\infty}\mathrm{d}\xi_p\int\frac{\mathrm{d}\Omega_{\vec{n}}}{\Omega_0}\int\frac{\mathrm{d}^dq_1\mathrm{d}^dq_2\mathrm{d}^dq_3}{(2\pi)^{3d}}G(\xi_p)\mathcal{T}_{\vec{q}_1}^{-1}\vec{n}\vec{q}_2G(\xi_p)\mathcal{T}_{\vec{q}_3}^{-1}\vec{n}(\vec{q}_1+\vec{q}_2+\vec{q}_3), \tag{4.24}$$

[17] See also Fig. on p. [55]250.

[18] In addition we assume that $\mathcal{T}$ is not proportional to the unity matrix in Keldysh space. Kamenev doesn't say this at this point, but assumes it further, in ([50]6.22).

[19] To save the letters of latin alphabet, we denote all our numerous actions with the letter S; but this doesn't mean that they are the same.

[20] Certainly, the result (4.21) can be also obtained in the momentum space, remembering about (4.20).

[21] The next (quadratic in q) term in (4.23) will not be zero, but we conceal ourselves with assumptions that this term will be very small.

where we used (4.22), and disregarded the difference in momentum arguments of GFs (taking into account that anyway we already have quadratic smallness in q). From (4.24) we conclude that

$$\mathrm{Sp}\left\{\left(G\mathcal{T}^{-1}\left[\xi,\mathcal{T}\right]_-\right)^2\right\} = \int\frac{\mathrm{d}^d k}{(2\pi)^d}\int\frac{\mathrm{d}^d q_1\mathrm{d}^d q_2}{(2\pi)^{2d}}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\left[\mathcal{T}_{\vec{q}_1}G_{\vec{p}-\vec{q}_1}\mathcal{T}^{-1}_{\vec{k}-\vec{q}_1}\left(\xi_{\vec{p}-\vec{k}}-\xi_{\vec{p}-\vec{k}-\vec{q}_2}\right)\mathcal{T}_{\vec{q}_2}G_{\vec{p}-\vec{k}-\vec{q}_2}\mathcal{T}^{-1}_{-\vec{k}-\vec{q}_2}\left(\xi_{\vec{p}}-\xi_{\vec{p}-\vec{q}_1}\right)\right].$$

In this expression the momentum $\vec{p}$ is much larger than all other momenta. The integrand is proportional to the square of the small momenta. In order not to introduce additional smalnesses, we assume that momenta of both GFs coincide (equal to $\vec{p}$), and integrate over $\vec{p}$. We get

$$\vec{\nabla}\mathcal{T}^{-1} \equiv -\mathcal{T}^{-1}\left(\vec{\nabla}\mathcal{T}\right)\mathcal{T}^{-1} \Longrightarrow \mathrm{Sp}\left\{\left(G\mathcal{T}^{-1}\left[\xi,\mathcal{T}\right]_-\right)^2\right\} = \frac{2\pi\nu\tau v_{\mathrm{F}}^2}{d}\sum_{s=\pm}\int\frac{\mathrm{d}^d k\mathrm{d}^d q_1\mathrm{d}^d q_2}{(2\pi)^{3d}}\left[\vec{q}_1\mathcal{T}_{\vec{q}_1}\left(\mathbb{1}+s\Lambda\right)\mathcal{T}^{-1}_{\vec{k}-\vec{q}_1}\vec{q}_2\mathcal{T}_{\vec{q}_2}\left(\mathbb{1}-s\Lambda\right)\mathcal{T}^{-1}_{-\vec{k}-\vec{q}_2}\right] =$$

$$= -\frac{2\pi\nu\tau v_{\mathrm{F}}^2}{d}\sum_{s=\pm}\mathrm{Sp}\left[\left(\vec{\nabla}\mathcal{T}\right)\left(\mathbb{1}+s\Lambda\right)\mathcal{T}^{-1}\left(\vec{\nabla}\mathcal{T}\right)\left(\mathbb{1}-s\Lambda\right)\mathcal{T}^{-1}\right] = \frac{4\pi\nu\tau v_{\mathrm{F}}^2}{d}\mathrm{Sp}\left\{\left[\left(\vec{\nabla}\mathcal{T}\right)\Lambda\mathcal{T}^{-1}\right]^2+\left(\vec{\nabla}\mathcal{T}\right)\left(\vec{\nabla}\mathcal{T}^{-1}\right)\right\} = \frac{2\pi\nu\tau v_{\mathrm{F}}^2}{d}\mathrm{Sp}\left[\left(\vec{\nabla}Q\right)^2\right].$$

We remember about the external field. Its linear contribution into the action is[22]

$$\mathrm{Sp}\left[G\mathcal{T}^{-1}V\mathcal{T}\right] = -i\pi\nu\,\mathrm{Sp}\left[\Lambda\mathcal{T}^{-1}V\mathcal{T}\right] = -i\pi\nu\,\mathrm{Sp}\left[QV\right]. \tag{4.25}$$

The last term in ([50]6.15) is the least clear to me. This term is stiff, it is independent on the impurity concentration, appears during taking the trace over frequencies (it should be taken the first). Since $\mathcal{T}$ are soft modes, they transfer small frequencies ω.[23] We assume that ME $\left\langle\vec{p},E\left|V_{cl,q}\right|\vec{p}-\vec{q},E-\omega\right\rangle$ is independent of $\vec{p}$ and E [that is, for the case of applied electric field (**??**) the results (4.26) and ([50]6.15) *are wrong*[24]]. We search for the terms integrals from which converge at large "main" energies E. For these terms we can ignore the presence of $\mathcal{T}$ in the leading order, so that

$$\mathrm{Sp}\left[\left(G\mathcal{T}^{-1}V\mathcal{T}\right)^2\right] \approx \mathrm{Sp}\left[(GV)^2\right] = \mathrm{Sp}\left[h_E\left(G_{\mathrm{R}}^2-G_{\mathrm{A}}^2\right)V_{cl}V_q+\left(G_{\mathrm{R}}^2+G_{\mathrm{A}}^2\right)V_{cl}^2+\left(G_{\mathrm{R}}-G_{\mathrm{A}}\right)^2V_q^2h_E^2+2G_{\mathrm{R}}G_{\mathrm{A}}V_q^2\right]. \tag{4.26}$$

Let us try to get the last term in ([50]6.15) from the first term in the rhs of (4.26) using (3.32).

$$\mathrm{Sp}\left[h_E\left(G_{\mathrm{R}}^2-G_{\mathrm{A}}^2\right)V_{cl}V_q\right] = \int\frac{\mathrm{d}^d p}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}h_E\left(G_{\mathrm{R}}^2-G_{\mathrm{A}}^2\right)\int\frac{\mathrm{d}^d q}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}V_{cl}(\vec{q},\omega)V_q(-\vec{q},-\omega)\propto$$
$$\propto \lim_{T\ll E^*\to\infty}\left(\nu_{E^*}+\nu_{-E^*}\right)\int\frac{\mathrm{d}^d q}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}V_{cl}(\vec{q},\omega)V_q(-\vec{q},-\omega) \approx 2\nu\int\frac{\mathrm{d}^d q}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}V_{cl}(\vec{q},\omega)V_q(-\vec{q},-\omega) \equiv \nu\,\mathrm{Sp}\left[V^T\sigma_1V\right], \tag{4.27}$$

so that Sp for the last term in the action ([50]6.15) has somewhat different sense than for other terms in ([50]6.15). I don't like the assumption that DoS is a constant, $\nu_E \equiv \nu = \mathrm{const}$. On the other hand, we remember about my old assumption which appeared during writing the Sec 1.5: if we say that $\int \mathrm{d}^d p/(2\pi)^d G_{\mathrm{R}}^2 = 0$, then we have to say that ν_E is a constant in the vicinity of $E=0$, asumingly, in the interval[25] $|E| \lesssim 1/\tau$. Correspondingly, in (4.27) we should by $E^*\to\infty$ understand $E^*\to 1/\tau$.

Consider other terms at large frequencies $E \gg T,\omega$ (where ω – are frequencies transfered by $\mathcal{T}$), $\Longrightarrow h_E^2 \approx 1$, so that the rest terms in (4.26) are equal to

$$\mathrm{Sp}\left[\left(G_{\mathrm{R}}^2+G_{\mathrm{A}}^2\right)\left(V_{cl}^2+V_q^2\right)\right] \propto \left(G_{\mathrm{R}}^E+G_{\mathrm{A}}^E\right)\Big|_{E=-\infty}^{E=\infty} = 0,$$

where we disregarded the frequency dependence of $V_{cl,q}$ (assuming that we are not interested in high-frequency perturbations,[26] so that $V_{cl,q}$ transfers small frequencies, just like $\mathcal{T}$.

Finally, if V corresponds to external electric field (**??**), equation (4.27) cancels the diamagnetic term.

So, we[27] got the desired action ([50]6.15). It is rather simple, so that we can calculate its second variation ([50]6.25,6.23).

[22] Eq. (4.25) is applicable only in case when V is independent on the modulus of momentum. For the case of electric field this means approximating $\vec{p}\approx p_{\mathrm{F}}\vec{n}$.

[23] More precisely, we assume that $\forall\vec{q},E\;\mathcal{T}_{E,E-\omega}(\vec{q})\to\mathbb{1}$ for $|\omega|\to\infty$. See also the text after ([52]47).

[24] In fact, when substituting (**??**) in ([50]6.15) we get divergence $\mathrm{Sp}\left\{V^T\sigma_1V\right\} = 2\int\frac{\mathrm{d}^d p}{(2\pi)^d}\left(\frac{ep_x}{2mc}\right)^2\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}A^E_{x,cl}A^{-E}_{x,q}$

[25] We implied that $T\tau\ll 1$, when we refused to consider stiff modes.

[26] High-frequency perturbations would lead to stiff modes from which we disowned from the very beginning.

[27] Apart from the lost coefficient which I order LATER.

4.5 Moral

Learning NLσM is useful because of the following reasons:

- This is an alternative way of calculating transport effects in diffusive conductors. Sure, not very convenient, but it is nice to hold it as a back-up.
- NLσM wakes up rememberings about relativistic quantum mechanics, and this is beautiful.
- NLσM allows understanding of NLσM-articles, and search for mistakes in them.
- Some people will think that you are very clever only because you use NLσM ☺.
- Some people will think that you are an idiot only because you don't know NLσM ☺.

Up to know[28] I have found no real calculational advantages of NLσM in comparison with the usual diagrammatics, but I've noticed disadvantages:

- NLσM is harder than the diagrammatics.
- Also in the diagrammatics the accuracy control is not ideal but in NLσM it is worse.
- How could one describe SOI in NLσM? I suspect – very uneasy. . .

[28] Hopefully – temporary, and everything changes when I read the articles cited by Lerner between Eqs. ([17]2-3).

Part II

Spin-orbit interaction

Chapter 5

Extrinsic spin-orbit: $V_{\rm so} \propto \vec{\sigma} \cdot [\vec{p} \times \vec{p}\,']$.

Literature: for general interest: cond-mat/0504175.

Note: Extrinsic SOI can not be ignored on the interface: there is a term in the Hamiltonian $\clubsuit \propto (\sigma_x \frac{\partial U}{\partial y} - \sigma_y \frac{\partial U}{\partial x})$, where U is a confining potential of the quantum well responsible for Rashba SOI $\propto \langle \frac{\partial U}{\partial z} \rangle$.

When we remmember about spin, we can generalize our non-interacting Hamiltonian to

$$\hat{H} = \sum_{\alpha=0}^{3} \sum_{\lambda,\lambda'} \varepsilon_\alpha(\lambda,\lambda') \psi^\dagger_\lambda \psi_{\lambda'}, \tag{5.1}$$

where λ is some quantum number *without* spin degree of freedom. Below we consider some particular cases of (5.1). According to Pis'ma v JETP **41**527,in the presence of magnetic impurities, a disorder averaging line is given by $f_{\alpha\beta\gamma\delta} = \frac{\delta_{\alpha\beta}\delta_{\gamma\delta}}{m\tau_0} + \frac{\vec{\sigma}_{\alpha\beta}\vec{\sigma}_{\gamma\delta}}{3m\tau_s}$. (In the same article, also expressions for cooperon and diffuson in this case are given.) See also appendix B from cond-mat/0402203.

In this section we consider the case when spin-orbital scattering is provided by impurities. A physical situation for this may be [56] "randomly placed heavy-ion impurities, which can simultaneously scatter electrons and flip their spin".

Let the potential of *one* impurity consist of 3 parts[1]:

$$U(\vec{r}) = \sigma^0 U_1(\vec{r}) + \vec{\sigma} \cdot \left[\vec{\nabla} U_3(\vec{r}) \times \vec{p}/p_F\right] + \vec{S}(\vec{r})\vec{\sigma}, \quad (\vec{\nabla} U_3)^\dagger = -\vec{\nabla} U_3. \tag{5.2}$$

Other ways of writing this Hamiltonian: cond-mat/0506589. How could one rewrite (5.3) in coordinate space?

According to [58], the impurity potential ME in the momentum representation is equal to

$$f_{\alpha\alpha'}(\vec{p},\vec{p}\,') = V\delta_{\alpha\alpha'} + V_S \vec{S}_i \vec{\sigma}_{\alpha\alpha'} - iV_{\rm so}[\vec{p} \times \vec{p}\,'] \cdot \vec{\sigma}_{\alpha\alpha'}.$$

Better see PRB**41**9548.

It might be described in [59] and in §140 of [60] how from this potential one comes to the one in [34], which is (in momentum representation):

$$U(\vec{p},\vec{p}\,') = U^*(\vec{p}\,',\vec{p}) = u\left[\sigma^0 + i\sqrt{\frac{\tilde{\tau}}{\tau_{\rm so}}}\vec{\sigma} \cdot [\vec{n} \times \vec{n}\,'] + \sqrt{\frac{\tilde{\tau}}{\tau_{\rm m}}}\vec{S}\vec{\sigma}\right], \tau_{\rm so}, \tau_{\rm m} \gg \tilde{\tau}, \tag{5.3}$$

where $\tilde{\tau}$ is defined according to (1.10), and $\vec{n} \equiv \vec{p}/|p|$ and $\vec{n}\,' \equiv \vec{p}\,'/|p'|$ denote the directions of momentum of an electron before and after its scattering on the impurity. The last term in (5.3) describes the presence of magnetic impurities; let us ignore it for the moment. One can note that, in 2D, extrinsic SOI conserves s_z.

From (5.3) it follows that the complete disorder field (from all impurities) will be

$$U(\vec{p},\vec{p}\,') = u\left[\sigma^0 + i\sqrt{\frac{\tilde{\tau}}{\tau_{\rm so}}}(\vec{\sigma},\vec{n},\vec{n}\,') + \beta\vec{S}\vec{\sigma}\right]\sum_a \exp\left[i(\vec{p}-\vec{p}\,')\vec{r}_a\right], \quad (\vec{\sigma},\vec{n},\vec{n}\,') \equiv \vec{\sigma} \cdot [\vec{n} \times \vec{n}\,'], \tag{5.4}$$

[1] H.-A. Engel with Rashba in cond-mat/0505535 say that $U_3 \equiv U_1$, so that apparently, $\tau_{\rm so}$ in (5.3) is fixed. Hans-Andreas Engel is not sure that (5.3) is equivalent to (5.2). He supposed that they are equivalent in case when Sherman function (see my Journal Club presentation here)

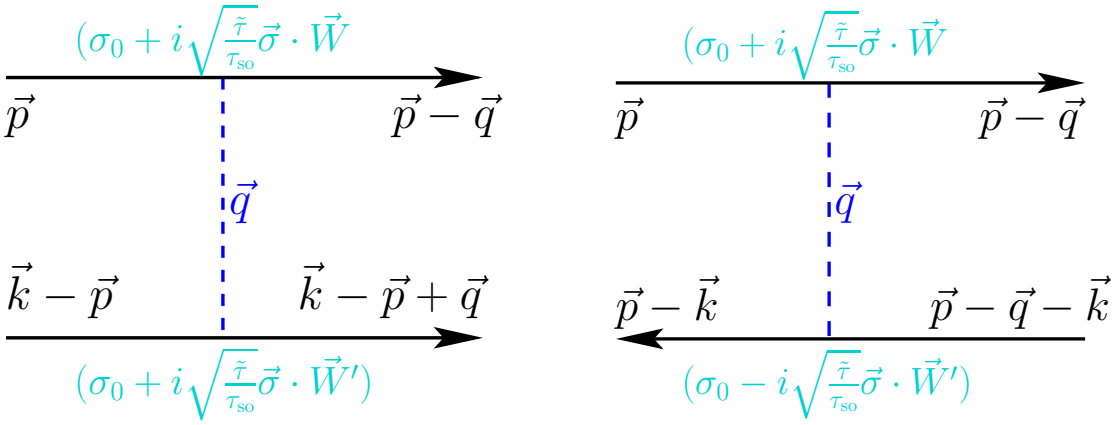


Figure 5.1: A block with one impurity averaging (dashed) line from fig. 2.2(a). $\vec W = [\vec p\times\vec p-\vec q] = -[\vec p\times\vec q]$, $\vec W' = [\vec k-\vec p\times\vec k-\vec p+\vec q] = [\vec k-\vec p\times\vec q] \approx \vec W$.

where $\sum_a e^{i\vec p\vec r_a}$ guarantees connection of dangling scattering lines while the averaging. Then the rest of the expression is just $u(\vartheta)$ in ([5]39.6), so that we deduce that for $G_{R/A}$ taking spin-orbital interaction into account results only in the renormalization of $\tilde\tau$. Below we immediately prove this statement rigorously.

Important: $\vec n$ and $\vec n\,'$ can be considered as totally uncorrelated. (because processes with a spin flip are relatively rare in comparison to "normal" scattering processes, that is, the ones without a spin flip)

Look on the diagrams with dashed lines in fig. 2.2(a). The integration is performed over the momentum passing through the all but one of these dashed lines. Let us consider one of many dashed line, see fig. 5.1. The vertices at the ends of the dashed line are just

$$1 + i\sqrt{\frac{\tilde\tau}{\tau_{so}}}\vec\sigma\cdot\vec W \equiv 1 + i\sqrt{\frac{\tilde\tau}{\tau_{so}}}(\vec\sigma,\vec p,\vec q) \equiv 1 + i\sqrt{\frac{\tilde\tau}{\tau_{so}}}e^{ijk}\sigma_i p_j q_k, \tag{5.5}$$

$$1 + i\sqrt{\frac{\tilde\tau}{\tau_{so}}}\vec\sigma\cdot\vec W' \equiv 1 + i\sqrt{\frac{\tilde\tau}{\tau_{so}}}(\vec\sigma,\vec k-\vec p,\vec q) \equiv 1 + i\sqrt{\frac{\tilde\tau}{\tau_{so}}}e^{lmn}\sigma_l(k_m-p_m)q_n. \tag{5.6}$$

The complete expression has to be integrated over the transfered momentum $\vec q$. The integration over its absolute value q must leave to some renormalization of τ_{so}. What is really important is the integration over the directions of $\vec q$. Instead of $\vec q$ I can integrate by $\vec W \approx \vec W'$. Due to the fact that

$$\forall k,n \quad \int d\vec W W_k = 0, \quad \int d\vec W W_k W_n \propto \delta_{kn},$$

spin-orbital part is not mixed with the usual one, so that there will be no linear in $\sqrt{\frac{\tilde\tau}{\tau_{so}}}$ terms in the resulting expression.

As a result, after the averaging we obtain an effective interaction line with an expression

$$\gamma^0 = \frac{1}{2\pi\nu}\left(\frac{1}{\tilde\tau}\sigma^0\sigma^{0\prime} + \frac{1}{\tau_{so}}\vec\sigma\vec\sigma'\right), \quad \tau_{so} \gg \tilde\tau. \tag{5.7}$$

Dealing with this expression, it makes sense to divide different spin components into a singlet and a triplet. That's because singlet and triplet are the eigenstates of the $\vec\sigma\vec\sigma'$ operator, so that for triplet $\gamma^0 = \gamma^0_T\sigma^0\sigma^{0\prime}$, and for singlet $\gamma^0 = \gamma^0_S\sigma^0\sigma^{0\prime}$, where

$$\gamma^0_T = \frac{1}{2\pi\nu}\left(\frac{1}{\tilde\tau} + \frac{1}{\tau_{so}}\right), \quad \gamma^0_S = \frac{1}{2\pi\nu}\left(\frac{1}{\tilde\tau} - \frac{3}{\tau_{so}}\right).$$

Repeating the derivation from sec. 1.4, we see that for a Green's function we always have $\vec\sigma\vec\sigma' = \sigma^0\sigma^{0\prime}$, so that the only change will be that $1/\tau$ will be substituted with $1/\tau_T = 1/\tilde\tau + 1/\tau_{so}$.

In other words: without spin-orbital interaction and for constant density of states we obtained from (1.12) that $2\pi\nu\tau\gamma^{(0)} = 2\pi\nu\tau n U_0^2 = 1$, see (1.10) and (1.15). In case when we have spin-orbital interaction, from (5.7) we have $2\pi\nu\tau_T\gamma^{(0)}_T = \frac{2\pi\nu\tau_T}{2\pi\nu}\left(\frac{1}{\tilde\tau} + \frac{1}{\tau_{so}}\right) = 1$, so that

$$\frac{1}{\tau_T} = \frac{1}{\tilde\tau} + \frac{1}{\tau_{so}}. \tag{5.8}$$

The cooperon's self energy is equal to $\frac{\gamma_T^0}{1-\gamma_T^0 X}$ for triplet and $\frac{\gamma_S^0}{1-\gamma_S^0 X}$ for singlet. Thus in the triplet case one obtains just a usual expression with τ substituted with τ_T given by (5.8), while in case of a singlet cooperon gains a mass: Seems that actually it is on the contrary: triplet gains mass, and singlet does not! See the text before ([10]2.14).

$$\sigma_{CT} = \frac{1}{2\pi\nu\tau_T^2}\frac{1}{Dq^2 - i\omega}, \quad \sigma_{CS} = \frac{1}{2\pi\nu\tau_S}\frac{1}{1-\frac{\tau_T}{\tau_S}\left[1-\tau_T(Dq^2-i\omega)\right]} \approx \frac{1}{2\pi\nu\tau_T^2}\frac{1}{Dq^2 - i\omega + \frac{4}{\tau_{so}}},$$

where we have taken into account that $\frac{\tau_S}{\tau_T^2} - \frac{1}{\tau_T} \approx \frac{4}{\tau_{so}}$.

As for the diffuson case, it seems that the only difference from cooperon's one will be the change of the sign of $\vec{W}'$ in fig. 5.1. Thus the expressions for the diffuson can be obtained from those for a cooperon by changing the sign in front of τ_{so}.

Chapter 6

Intrinsic spin-orbit

Literature: [16] and [61] are my favorite; despite that [62] is a bad book written on purpose to confuse people, I still sometimes open it if I don't find the information elsewhere. Here is the Hamiltonian which we are interested in:

$$\hat{H} = \frac{\hat{p}^2}{2m} + a(\sigma_1\hat{p}_y - \sigma_2\hat{p}_x) + b(\sigma_1\hat{p}_x - \sigma_2\hat{p}_y) + u(\vec{r}), \quad \hat{\vec{p}} \equiv -i\hbar\vec{\nabla}, \quad a, b \in \mathfrak{R}. \tag{6.1}$$

6.1 SOI and disorder

See definitions in [20]. In the denominator of a GF, we ignore all terms which are higher than linear in ξ. Thus if we considered cubic SOI (instead of the linear one), then we should have substituted $p^3 \to 2m(E_F + \xi)p_F$. Thus it can happen that the cubic SOI brings different physics than the considered here linear SOI. However, higher-power terms (e.g., $\propto p^5$) are similar to the cubic one, so considering them is senseless.

Rashba Hamiltoniam in the second quantization can be found in PRL**91**226803(2005); on the lattice – in cond-mat/0504218. The exact eigensystem of the Hamiltonian (6.1) with the magnetic field is claimed to be found in [63]. Also take a look on how SOI is affected by the interaction: PRB**77**233310. Let us define the s.c. SOI lengths $\lambda_{1,2}$ [0409054][1]:

$$\frac{\hbar}{\lambda_{1,2}} = m(a \pm b), \tag{6.2}$$

Without loss of generality (since the contemporary sign change of a and b does not change the conductivity, which we are interested in[2]) we assume that $a + b > 0$. Later, in (6.19) we will expand our GFs in small parameters x and δ defined in (6.3) and in (6.19) using $a - b = \mathrm{sign}(a-b)x\sqrt{1-\delta}/(2p_F\tau)$; so we will have to consider two cases $\mathrm{sign}(a-b) = \pm 1$ separately.

The amplitude of the SOI can be characterised with a dimensionless parameter x, as well as with the characteristic momentum p_S or energy Δ_0:

$$p_S = m\sqrt{a^2+b^2} = \frac{x\hbar}{2l}, \quad \Delta_0 = 2p_F\sqrt{a^2+b^2} = \frac{x\hbar}{\tau}. \tag{6.3}$$

Note that $\hat{H}'$ is invariant with respect to the time reversal: $K\hat{H}'K^{-1} = \hat{H}'$, where K is defined in Sec. **??**. The advantage of this new (rotated by $\pi/4$) CS is that conductivity tensor is diagonal in it.[3] Formally applying Legendre transformation (just like in classical mechanics), we arrive to Lagrangian

$$L = \frac{p^2(\vec{v})}{2m} - u(R^{-1}_{\pi/4}\vec{r}), \quad \vec{p}(\vec{v}) = m\vec{v} - \frac{e}{c}\vec{\tilde{A}}.$$

[1] See other SOI lengths in [64].

[2] This is true even without averaging over the disorder. In fact, let us first build *unaveraged* GF operators $\hat{G}_{R/A}$ using the Hamiltonian (6.2) with $a = b = 0$, and make perturbation expansion of (8.7) [including the SOI-dependent part of the vertices $\hat{v}_{\alpha,\beta}$] in powers of some SOI amplitude [which should change sign when both a and b do that]. Let us consider odd terms of this expansion. The expression under Sp in (8.7) will be proportional to some linear combination of σ_1 and σ_2, so that $\mathrm{Sp} = 0$ for arbitrary odd (in SOI amplitude) term. (Here we have used the fact that our Hamiltonian and vertices $\hat{v}_{\alpha,\beta}$ do not contain terms $\propto \sigma_3$.)

[3] I realize it in the end of the calculation. I would like to prove this using (6.9) and the explicit form of the velocity operator (6.8), STOP but I can't.

where $\vec{A}$ is defined in (6.8). Analogously to the spirality operator in QED,[4] our "spirality" operator $M_{\vec{p}}$ (denoted by $\hat{s}$ in [20]) commutes with the Hamiltonian in the absence of disorder. The Hamiltonian has a specific symmetry with respect to rotation[5] by $\pi/2$:

$$\hat{H}(\vec{p},\vec{\sigma}) = \hat{H}(-\vec{p},-\vec{\sigma}) = \hat{H}(R_{\pm\pi/2}\vec{p}, R_{\pm\pi/2}\vec{\sigma})\Big|_{b\to -b}, \tag{6.5}$$

or, in other words, Rashba SOI is invariant under arbitrary rotation [66], while Dresselhaus SOI is not; the rotation by $\pi/2$ is special [66].

Note the significant difference of (6.1) from (5.3): here spin-orbital scattering is not disordered, and the disorder potential is considered as spin-independent. Also division to singlet and triplet does not work: it is impossible to diagonalize spin-orbit term in the Hamiltonian independently of the momentum value. For the same reason $G_{R/A}$ are essentially non-diagonal matrices in spinor space.

An expression for the electric current can be derived in two[6] ways. At first, from the charge conservation law $\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$. The second way is to substitute velocity operator (1.5) into the classical formula $\vec{j} = en\vec{v}$, that is, $\hat{\vec{j}} = ie[\hat{H}, x]/\hbar = e\vec{\nabla}_{\vec{p}}\hat{H}$, so that a state $|\psi\rangle$ would produce the current density $\vec{j} = e\psi^\dagger(\vec{r})\hat{\vec{v}}\psi(\vec{r})$. However, this quantity will be non-hermitian. Without being upset by this fact, we just take its hermitian part, and obtain the usual formula for the current density and its operator.

Let us try to follow the first way in derivation of the current, trying to obtain it from the conservation law[7]

$$i\hbar\frac{\partial}{\partial t}(\psi^\dagger\sigma_\gamma\psi) = -(\psi^\dagger\hat{H})\sigma_\gamma\psi + \psi^\dagger\sigma_\gamma(\hat{H}\psi) = \frac{\hbar^2}{2m}\vec{\nabla}\left[(\vec{\nabla}\psi^\dagger)\sigma_\gamma\psi - \psi^\dagger\sigma_\gamma\vec{\nabla}\psi\right] -$$
$$-\frac{i}{m}\left(\frac{\psi^\dagger\sigma_\gamma\sigma_1\psi_y}{\lambda_2} + \frac{\psi_y^\dagger\sigma_1\sigma_\gamma\psi}{\lambda_2} - \frac{\psi^\dagger\sigma_\gamma\sigma_2\psi_x}{\lambda_1} - \frac{\psi_x^\dagger\sigma_2\sigma_\gamma\psi}{\lambda_1}\right), \quad \gamma = 0\ldots 3. \tag{6.6}$$

For $\gamma = 0$, we obtain the charge conservation law with the charge current given by[8]

$$\text{In the rotated basis:} \quad \vec{j} = \frac{ie\hbar}{2m}\left[\left(\vec{\nabla}\psi^\dagger\right)\psi - \psi^\dagger\vec{\nabla}\psi + 2\frac{ie}{\hbar c}\vec{A}_\omega\psi^\dagger\psi\right] + \frac{e}{m}\left(-\frac{\psi^\dagger\sigma_2\psi}{\lambda_1}, \frac{\psi^\dagger\sigma_1\psi}{\lambda_2}, 0\right), \tag{6.7}$$
$$\text{In the original basis:} \quad \vec{j} = \frac{ie\hbar}{2m}\left[\left(\vec{\nabla}\psi^\dagger\right)\psi - \psi^\dagger\vec{\nabla}\psi + 2\frac{ie}{\hbar c}\vec{A}_\omega\psi^\dagger\psi\right] + e\left(b\psi^\dagger\sigma_1\psi - a\psi^\dagger\sigma_2\psi, a\psi^\dagger\sigma_1\psi - b\psi^\dagger\sigma_2\psi, 0\right).$$

Comparing it with (13.8), we see that to calculate the current we can use the usual charge current formula [20] with a fictitious[9] vector potential[10]

$$\vec{\tilde{A}} = -\frac{c}{e}\left(-\frac{\sigma_2}{\lambda_1}, \frac{\sigma_1}{\lambda_2}, 0\right) = \frac{mc}{\hbar e}\left[(a+b)\sigma_2, (b-a)\sigma_1, 0\right], \quad \hat{\vec{v}} = \frac{i}{\hbar}[\hat{H}, \vec{r}] = \frac{\hbar\hat{\vec{p}}}{m} - \frac{e}{mc}\left(\vec{A} + \vec{\tilde{A}}\right), \tag{6.8}$$
$$\hat{H} = \frac{m\hat{v}^2}{2} - m(a^2+b^2) + U(\vec{r}), \quad R_{\pi/2}\vec{v} = \begin{pmatrix} v_y \\ -v_x \end{pmatrix} = R_{\pi/2}\left(\frac{\vec{p}}{m} + \vec{A}\right) + \vec{\tilde{A}}\Big|_{\sigma\to R_{\pi/2}\sigma}. \tag{6.9}$$

---

[4] а) Следует иметь в виду, что, строго говоря, спиральность и киральность – это не одно и то же, см. стр. [65]110. б) STOP Проверить, что спиральность электрона сохраняется. ☞ From QED we know that spirality is a projection of spin onto momentum. However, [20](**??**) seems a different quantity. The common thing between these two definitions is that spirality remains invariant under the time-reversal:

$$\hat{M}_{\vec{p}} = \sigma_2\left[\hat{M}_{-\vec{p}}\right]^T\sigma_2. \tag{6.4}$$

This property is valid not only for any bilinear form $A(\vec{p},\vec{\sigma})$, but also for many polynomials, e.g., cubic Rashba and Dresselhaus. [BTW, this is why quadratic (in momentum) SOI is impossible.]

[5] I can consider such a rotation as a unitary transformation. The corresponding unitary matrices (in both momentum and spin space) are given by (13.40) at $\varphi = \pi/4$.

[6] There is a third way described in §[60]115 – see the 3th footnote before Eq. (13.8): "The variation of the Hamiltonian with respect to the electromagnetic vector potential $\vec{A}$ gives electric current operator." This way has (apparently) been used in [67] for the spin current, and no additional **rot**-term has been obtained there. Finally, I want derive the current operator from the Nöther theorem in the QFT-fashion of [68].

[7] I used brackets in (6.6) to emphasize that in the expression $(\psi^\dagger\hat{H})\sigma_\gamma\psi$, the differentiating operator $\hat{\vec{p}}$ in $\hat{H}$ act on $\psi^\dagger$, *not* on $\sigma_\gamma\psi$. By definition, $\psi^\dagger\hat{H} \stackrel{\text{df}}{=} (H\psi)^\dagger$.

[8] STOP And what about the case of random (extrinsic) scattering from the 5th chapter? Is the current there modified as well?

[9] From (6.8) we see that the velocity operator is affected by the SOI. In ([0603144]3) it is claimed that the coordinate operator is changed as well.

[10] This permits us using formulas like (3.38), derived for the case of no spin-orbit interaction, by replacing $\vec{p} \to \vec{p} - \frac{e}{\hbar c}\vec{\tilde{A}}$. Note that $\vec{\tilde{A}}$ gets renormalized according to (7.13). Note also the importance of the inversion of the order of quantum numbers $\lambda \equiv (\vec{r}, t)$ and $\lambda' \equiv (\vec{r}', t)$ in (9.4).

Note that we don't get the **rot**-term mentioned in the 6th footnote above; if we are interested in this term, we should manually derive it from the Dirac equation; it is an "external ingredient" of the same type as SOI.

From (6.9) one obtains BC[11] for the connection of WF in two neighboring materials with different SOI amplitudes and disorder. Using the second expression for $\hat{H}$, we integrate SE $\hat{H}\psi = E\psi$ over the infinitesimal neighbourhood of the border, and see that a generalized BC is that the spinor $\hat{\vec{v}}\psi(\vec{r})$ must be a continous function of the coordinate. Without SOI, this corresponds to the usual text-book requirement that $\psi'(\vec{r})$ is continuous.

6.2 When $a = \pm b$

In the special case $a = \pm b$, the Hamiltonian $\hat{H}'$ is diagonalized by the unitary transformations. In the original CS:

$$
\begin{aligned}
a = b \Rightarrow & U_1 V_s' U_1^\dagger = -\sqrt{2}a(p_x' + p_y')\sigma_3, \quad U_1 \vec{v}\,' U_1^\dagger = -\sqrt{2}a \begin{pmatrix} \sigma_3 \\ \sigma_3 \end{pmatrix}, \\
a = -b \Rightarrow & U_2 V_s' U_2^\dagger = -\sqrt{2}a(p_y' - p_x')\sigma_3, \quad U_2 \vec{v}\,' U_2^\dagger = \sqrt{2}a \begin{pmatrix} \sigma_3 \\ -\sigma_3 \end{pmatrix}, \\
& U_1 = (U_1^\dagger)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{i\pi/4} \\ 1 & e^{i\pi/4} \end{pmatrix}, \quad U_2 = (U_2^\dagger)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -e^{-i\pi/4} \\ 1 & e^{-i\pi/4} \end{pmatrix}.
\end{aligned}
$$

Especially nice this looks in the rotated CS:

$$
\begin{aligned}
a = b \Rightarrow & U_1 V_s U_1^\dagger = -2ap_y\sigma_3, \quad U_1 \vec{v} U_1^\dagger = \begin{pmatrix} \frac{p_x}{m} & \\ & \frac{p_y}{m} - 2a\sigma_3 \end{pmatrix}, \\
a = -b \Rightarrow & U_2 V_s' U_2^\dagger = -2ap_x\sigma_3, \quad U_2 \vec{v} U_2^\dagger = \begin{pmatrix} \frac{p_x}{m} - 2a\sigma_3 & \\ & \frac{p_y}{m} \end{pmatrix}, \\
& U_1 = (U_1^\dagger)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad U_2 = (U_2^\dagger)^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.
\end{aligned}
\tag{6.10}
$$

In Sec. 7.5 and 8.2 we see that this case leads to the same results, as in the absence of SOI.

6.3 Spin current (non) conservation

Let us rewrite (6.6) for the z spin component:

$$
i\hbar \frac{\partial}{\partial t}(\psi^\dagger \sigma_z \psi) = \frac{\hbar^2}{2m} \vec{\nabla} \left[(\vec{\nabla}\psi^\dagger)\sigma_z \psi - \psi^\dagger \sigma_z \vec{\nabla}\psi \right] - \frac{1}{m} \left[(\psi_y^\dagger \sigma_2 \psi - \psi^\dagger \sigma_2 \psi_y)/\lambda_2 + (\psi_x^\dagger \sigma_1 \psi - \psi^\dagger \sigma_1 \psi_x)/\lambda_1 \right]. \tag{6.11}
$$

Since the right part of (6.11) can not be written as a divergence of any vector, s_z is not conserved[12] (and neither s_x and s_y do), so that we have to use the second way for the derivation of its current[13]:

$$
[\Delta, \vec{r}\,] = 2\vec{\nabla}, \quad \hat{\vec{j}}_{s_i} = \hat{s}_i \hat{\vec{v}}, \quad \hat{\vec{v}} = \dot{\vec{r}} = \frac{i}{\hbar}[\hat{H}, \vec{r}\,] = -\frac{i\hbar}{m}\vec{\nabla} + \frac{1}{m}\left(\frac{\sigma_1 \hat{\vec{y}}}{\lambda_2} - \frac{\sigma_2 \hat{\vec{x}}}{\lambda_1} \right) - \frac{e}{mc}\vec{A}, \tag{6.12}
$$

$$
\begin{aligned}
& \hat{\vec{j}}_{s_i'} = -\frac{\sigma_i \hbar}{2m} \left[i\hbar\vec{\nabla} - \left(\frac{\sigma_1 \hat{\vec{y}}}{\lambda_2} - \frac{\sigma_2 \hat{\vec{x}}}{\lambda_1} \right) + \frac{e}{mc}\vec{A} \right], \quad \hat{\vec{j}}_{s_i'}(\vec{r}) = \hat{\psi}^\dagger(\vec{r}) \hat{\vec{j}}_{s_i'} \hat{\psi}(\vec{r}), \\
& \hat{\vec{j}}_{s_i}(\vec{r}) = \frac{1}{2}\left[\hat{\vec{j}}_{s_i'}(\vec{r}) + \hat{\vec{j}}_{s_i'}^\dagger(\vec{r}) \right] = \frac{1}{2}\left\{ \hat{s}_i, \hat{\vec{v}} \right\} = \\
& = -\frac{i\hbar^2}{4m} \left[\hat{\psi}^\dagger(\vec{r})\sigma_i \vec{\nabla}\hat{\psi}(\vec{r}) - (\vec{\nabla}\hat{\psi}^\dagger(\vec{r}))\sigma_i \hat{\psi}(\vec{r}) - \frac{2ie}{\hbar c}\vec{A}\psi^\dagger(\vec{r})\sigma_i \psi(\vec{r}) \right], \; i = 1 \div 3.
\end{aligned}
\tag{6.13}
$$

$$
\text{In the (usual) } \sigma\text{–basis, } b = 0: \quad \hat{\vec{v}} = -\frac{i\hbar}{m}\vec{\nabla} + a(\sigma_1 \hat{\vec{y}} - \sigma_2 \hat{\vec{x}}) - \frac{e}{mc}\vec{A}, \tag{6.14}
$$

[11] a) См. письма в ЖЭТФ т. 30 стр. 574. b) A consideration of such a boundary must involve additional (extrinsic) SOI effects, see the beginning of Sec. 5.

[12] That is why it is unclear, how spin current should be defined, see PRL**96**076604.

[13] I've written it for more general case when we have $\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$ instead of p^2 in (6.1). См. [60], §111, стр. 530-531.

We see that spin-orbital terms in the Hamiltonian make no effect on the expression for the spin current[14]. It is easy to obtain an expectation valuse of the spin current from (6.13): just remove operator's hats, integrate over coordinate and take a spin trace. If the integration is done over the whole space, then the second term in (6.13) is obtained from the first one using integration by parts.

Now we can rewrite (6.11) as

$$\dot{s}_z(\vec{r}) + \operatorname{div} \vec{j}^{s_z} = \frac{2}{\hbar}\left(\frac{j_x^{s_x}}{\lambda_1} + \frac{j_y^{s_y}}{\lambda_2}\right).$$

Thus s_z is not conserved[15], but still expressed in terms of spin currents. This is not true for s_x and s_y.

In practical calculation of s_z current, one should use the usual charge current formula [20] with the removed diamagnetic term $\vec{A} \to 0$ and e substituted by $\sigma_3\hbar/2$ (with a Sp operator added). As for $\vec{A}$, we can set it to 0 because, if we calculate *averaged* over impurities spin-current, the diamagnetic term is given by some combination of terms $\mathrm{Sp}\,[\sigma_3 G_{\mathrm{R/A}}]$ which both are equal to zero due to the fact that averaged $G_{\mathrm{R/A}}$ depend linearly on σ_1 and σ_2 and does not depend on σ_3.

Some papers contradict this reasoning: [70], cond-mat/0410607.

6.4 The eigensystem and Green functions

Without the disorder part, the eigensystem of the Hamiltonian (6.2) is given by[16,17,18]:

$$\varphi_{\vec{p}s}(\vec{r}) = \frac{e^{i\vec{p}\vec{r}}}{\sqrt{2V}}\begin{pmatrix} s\sqrt{i\frac{p_-}{\tilde{p}}} \\ 1 \end{pmatrix}, \quad \int_V \mathrm{d}^d r \left[\varphi_{\vec{p}s}(\vec{r})\right]^\dagger \varphi_{\vec{p}s}(\vec{r}) = 1, \quad E_{\vec{p}s} = \frac{\hbar^2}{2m}p^2 + \frac{s\Delta_{\vec{p}}}{2}, \quad s = \pm 1, \tag{6.15}$$

$$p_\pm = \frac{p_x}{p_S\lambda_1} \pm i\frac{p_y}{p_S\lambda_2}, \quad \tilde{p} = |p_\pm| \stackrel{\mathrm{df}}{=} \tilde{n}p. \tag{6.16}$$

Using Lehmann representation (3.29) we obtain expression for the $G^{(0)}_{\mathrm{R/A}}$:

$$G^{(0)}_{\mathrm{R/A}}(\vec{r},\vec{r}';E) = \frac{1}{2V}\sum_{\vec{p}s}\frac{e^{i\vec{p}(\vec{r}-\vec{r}')}}{E - E_{\vec{p}s} + E_{\mathrm{F}} \pm i\varepsilon}\begin{pmatrix} 1 & s\sqrt{i\frac{p_-}{\tilde{p}}} \\ s\sqrt{-i\frac{p_+}{\tilde{p}}} & 1 \end{pmatrix},$$

$$G^{(0)}_{\mathrm{R/A}}(\vec{p},E) = \frac{1}{2}\left\{\frac{\sigma_0 + M_{\vec{p}}}{E - E_{\vec{p}+} + E_{\mathrm{F}} \pm i\varepsilon} + \frac{\sigma_0 - M_{\vec{p}}}{E - E_{\vec{p}-} + E_{\mathrm{F}} \pm i\varepsilon}\right\} = \tag{6.17}$$

$$= \frac{1}{2}\left\{\frac{\sigma_0 + M_{\vec{p}}}{E - \frac{(p+p_0)^2}{2m} + E_{\mathrm{F}} + \frac{p_0^2}{2m} \pm i\varepsilon} + \frac{\sigma_0 - M_{\vec{p}}}{E - \frac{(p-p_0)^2}{2m} + E_{\mathrm{F}} + \frac{p_0^2}{2m} \pm i\varepsilon}\right\} =$$

$$= \frac{E - \xi_p \pm i\varepsilon + \frac{\Delta}{2}M_{\vec{p}}}{(E - \xi_p - \frac{\Delta}{2} \pm i\varepsilon)(E - \xi_p + \frac{\Delta}{2} \pm i\varepsilon)}, \quad \varepsilon = +0, \quad \hbar p_0 = p_S\frac{\tilde{p}}{p} = p_S\tilde{n}, \tag{6.18}$$

where $M_{\vec{p}}$ can be considered as a generalization of the spirality (=helicity) operator defined in ([24]5). Its generalization for the case MF is done in ([0508681]3).

The spirality is connected with the SOI-induced spectrum splitting as [20]

$$\Delta_{\vec{p}} = \frac{p}{p_{\mathrm{F}}}\Delta_0\sqrt{1 + \delta\cos(2\varphi)}, \quad !\hat{s} \equiv M = \frac{\sigma_1 \sin\varphi\sqrt{1-\delta}\,\mathrm{sign}(a-b) - \sigma_2\cos\varphi\sqrt{1+\delta}}{\sqrt{1+\delta\cos 2\varphi}}, \quad \delta = \frac{2ab}{a^2+b^2}, \quad -1 \le \delta \le 1. \tag{6.19}$$

where $\mathrm{sign}(a-b) \equiv 1$ if we additionally assume[19] that $|a| > b$ in [20](**??**). The parameters x and δ are *correct* expansion parameters[20] for σ_{xx} and σ_{yy} in the sense that, for arbitrary function $f(\Delta_{\vec{p}})$, [in other words, the expansion in (x,δ) is

[14] An interesting expression of spin current through the distribution function: ([69]7).

[15] Spin-orbital term does not violate time-inversion symmetry. In round quantum dots of VG it violates rotational symmetry, but in my dirty metal I don't have it anyway, so basically I loose no integrals of motion.

[16] In the original, not turned basis, the energy spectrum $E_{\vec{p}s}$ and eigenvectors $\varphi_{\vec{p}s}$ look differently from (6.15). Take a look on `chiral.tm`, [71], and [49].

[17] Note that in the special case $a = b$ the spin part of the WF can be made separate from its orbital part (like in the absence of SOI). One can see that after making discrete variable substitution $s = s'\,\mathrm{sign}\,p_x$. Then, e.g., $E_{\vec{p}s'} = \frac{\hbar^2}{2m}p^2 + s'|a|p_x$. [This substitution is, actually, compulsory: without it the Hamiltonian would depend on badly defined operator $|\hat{p}_x|$.] A similar situation must happen also in case when $a = -b$.

[18] I should carefully realize that the basis in (6.15) is invariant under the time reversal transformation.

[19] See the discussion before (6.3).

[20] $x/2$ and δ are called correspondingly "quadratisches Mittelwert" and "harmonisches Mittelwert" of x_a and x_b.

uniform]

$$\lim_{x\to 0}\frac{\partial^n}{\partial x^n}\lim_{\delta\to 0}\frac{\partial^m}{\partial \delta^m}f(\Delta_{\vec{p}})=\lim_{\delta\to 0}\frac{\partial^m}{\partial \delta^m}\lim_{x\to 0}\frac{\partial^n}{\partial x^n}f(\Delta_{\vec{p}}), \tag{6.20}$$

which is *not true* for the expansion in x_a and x_b, see the p. 65. In addition, it is nice that x and δ have physical sense: the amplitude of the SOI and the strength of its anisotropy.

Note that $\clubsuit K G_{\mathrm{R}}(\vec{p})K^{-1}=G_{\mathrm{A}}^{T}(\vec{p})$, where K is the time inversion operator from Sec. 6. In case[21] $b=0$, eigenvalues are the same in σ- and σ-basis. (since in this case they are independent on the direction of $\vec{k}$); eigenvectors in σ-basis can be obtained from (6.15) by substituting $\sqrt{i}\to i$; as for M and GF, they can be obtained from (6.17) and (6.3) by substituting $\sigma_{12}\to\sigma_{12}$.

$$\text{A useful identity in 2D}:\arg(\vec{p}+\vec{q})-\arg\vec{p}=\frac{q}{p}\sin(\arg\vec{q}-\arg\vec{p})+O\left[\left(\frac{q}{p}\right)^2\right],\quad q\ll p.$$

Note offdiagonal elements of (6.17) which are proportional to $k_{\pm}$, which correspond to the $L_z=\pm 1$ total momentum state. In fact somehow I could say that $(G_{\mathrm{R/A}})_{\uparrow\uparrow}$ and $(G_{\mathrm{R/A}})_{\downarrow\downarrow}$ correspond to the triplet with $S_z=\pm 1$. It would be nice to draw the inverse Fourier images of $k_{\pm}/k$ in real space.

In case when $a\neq b$ this defines two Fermi surfaces (lines in our 2D system) instead of one, which are nor circles, nor ellipses [72].

Let us now calculate the series for the self energy in the self-consistent Born approximation. It is given by the series of diagrams (1.7). Only diagrams without intersecting dashed lines are considered in (1.7) because, according to [5] and Sec. 1.4, every intersection results in the reduction of an integration volume in the momentum space, leading to the smallness of the order of $(p_{\mathrm{F}}l)^{-1}\ll 1$. Then, in the approximation of the constant density of states all terms in (1.7) but the first one are equal to zero due to the analytical properties of $G_{\mathrm{R/A}}$: indeed, every such a term contains at least one multiplier $\propto \nu\int_{-E_{\mathrm{F}}}^{\infty}\mathrm{d}\xi G_{\mathrm{R/A}}^2(\vec{p},E)\approx\nu\int_{-\infty}^{\infty}\mathrm{d}\xi G_{\mathrm{R/A}}^2(\vec{p},E)=0$. [the quality of this approximation is $\sim(E_{\mathrm{F}}\tau)^{-1}$, as it is discussed on p. 51.] Calculating the first term in (1.7), we see that the spin dependence from (6.17) is smeared due to the integration over the directions of momentum. In other words, in case when $|a|=|b|$, the presence of spin-orbital term in the Hamiltonian (6.1) *does not affect* $\tau=\tilde{\tau}\equiv(2\pi\nu n U_0^2)^{-1}$.

Now an averaged Green function is equal to[22]

$$G_{\mathrm{R/A}}=\frac{1}{{G_{\mathrm{R/A}}^{(0)}}^{-1}\pm\sigma_0\frac{i}{2\tau}}. \tag{6.21}$$

One can check that

$$\forall a,b,c\in\mathbb{R}\quad (a\sigma_0+bM)^{-1}=\frac{a\sigma_0-bM}{a^2-b^2},\quad (a\sigma_0+bM+c\sigma_3)^{-1}=\frac{a\sigma_0-bM-c\sigma_3}{a^2-b^2-c^2}. \tag{6.22}$$

From (6.21) one obtains an expression for the averaged Green function. Just like in the usual case (when there is no Rashba term), the expression for it can be obtained by substituting $\varepsilon=+0$ with $\frac{1}{2\tau}$:

$$G_{\mathrm{R/A}}(\vec{p},E)=\frac{E-\xi(p)\pm\frac{i}{2\tau}+\frac{\Delta}{2}M}{[E-\xi(p)-\frac{\Delta}{2}\pm\frac{i}{2\tau}][E-\xi(p)+\frac{\Delta}{2}\pm\frac{i}{2\tau}]}, \tag{6.23}$$

$$G_{\mathrm{R/A}}(\vec{p})\sigma_3=\sigma_3 G_{\mathrm{R/A}}(-\vec{p}),\quad G^{*}_{\mathrm{R/A}}(p_x,p_y)=G_{\mathrm{A/R}}(-p_y,-p_x),\quad \sigma_2 G_{\mathrm{A}}^{T}(-\vec{p})\sigma_2=G_{\mathrm{A}}(\vec{p}) \tag{6.24}$$

where[23] by $g_{\mathrm{r/a}}$ we implied "usual" averaged Green functions (1.9).

One can insert the resulting $G_{\mathrm{R/A}}$ from (6.23) into the self-consistent Born approximation, to check, if it is really self-consistent. The answer is "yes" within the approximation $\Delta/E_{\mathrm{F}}\ll 1$. In fact, if we rewrite (1.7) as

$$\mp\frac{1}{2\tau}=\Im\int_{-\infty}^{\infty}\mathrm{d}\xi G_{\mathrm{R/A}}(\xi)-\Im\int_{-\infty}^{-E_{\mathrm{F}}}\mathrm{d}\xi G_{\mathrm{R/A}}(\xi)\equiv I-I',$$

[21] When both $a,b\neq 0$, in the original basis $M=(ap_y+bp_x)\sigma_1/\Delta-(ap_x+bp_y)\sigma_2/\Delta$ with $\Delta^2=(a^2+b^2)p^2+4abp_xp_y$.

[22] During the calculation of self energy in (6.21) I approximated $\alpha\tilde{k}\approx\alpha p_{\mathrm{F}}$. Also without this approximation the self energy would not be just $\sigma_0/2\tau$, like in (6.21), but $\tau\neq\tilde{\tau}$. In principle one needs not doing this approximation; if we don't employ it, then in 2D we substitute $\xi\to p^2/(2m)-E_{\mathrm{F}}$, so that (for $|\beta|=1$)

$$\clubsuit\,\Im\sigma_{\mathrm{R}}=-\frac{\sigma^0}{16\pi\tau}\left[J(\Delta)-J(-\Delta)\right],\quad J(\Delta)=\int_0^{\infty}\mathrm{d}p\frac{p}{\left(E-\frac{p^2}{2m}+E_{\mathrm{F}}-\alpha p\right)^2+\frac{1}{4\tau^2}}$$

STOP suspect that there is + instead of – in the last eq. The result for the averaged $G_{\mathrm{R/A}}$ will still be (6.23), but now with $\tau\neq\tilde{\tau}$.

[23] Note that this result (checked by myself) is in contradiction with PRB64144423.

then one can estimate $I' \sim I/(E_F\tau)$. This looks like a small adjustment of τ, so that it is not important. However, the same reasoning is true for the integrals (**??**),**??**, and this is equivalent to say that cooperon and diffuson have a (real!) mass of the order of $1/(E_F\tau^2)$. This can be taken into account by keeping in mind that there is always a small $\sim 1/(E_F\tau^2)$ cut-off for the frequency. For example in the result (7.9) we have to remember that in reality always $|\omega| \gtrsim 1/(E_F\tau^2)$, and this determines the order of limits in (7.9): first one has to take $\lim_{x\to 0}$ and only afterwards[24] $\lim_{\omega\to 0}$.

This discussion can be expanded for quantities like $\int \frac{d^2p}{(2\pi)^2} g_r(p,E)g_a(p,E-\Delta)$ which occur in the calculation of (**??**): whenever we consider an integral of this type, we do it with a precision $(E_F\tau)^{-1} \ll 1$ given that all other energy scales in the system like Δ and temperature are smaller than $1/\tau$. The assumption $\Delta \ll \tau^{-1}$ permits me to treat Δ as a usual frequency variable.

Then cooperon and diffuson are calculated almost like in the spinless case. One notes that $\vec{n} \approx \vec{p}/|p|$ for $G_{R/A}(\vec{p})$ and $\vec{n} \approx -\vec{p}/|p|$ for $G_{R/A}(\vec{q}-\vec{p})$.

6.5 Dealing with SOI without SCBA

The calculation of the averaged GF from the previous section can be implemented in a more elegant manner. Let us suppose the SOI-part of the Hamiltonian is small so that we can consider it as a perturbation $\delta\hat{H}$. Like we did it before in (**??**), let us then use perturbation theory expansion for G_R. Let us at first consider the situation, when the Dresselhaus contribution to the SOI is absent, so that $b = 0$ in the Hamiltonian (6.1). Then the infinite perurbation series sums up into

$$G_R = G_R^{(0)} \sum_{n\geq 0} \left(\delta\hat{H} G_R^{(0)}\right)^n = \left[\left(G_R^{(0)}\right)^{-1} - apM\right]^{-1} = \frac{\left(G_R^{(0)}\right)^{-1} + apM}{\left(G_R^{(0)}\right)^{-2} - a^2p^2}, \tag{6.25}$$

where $G_R^{(0)}$ stands for the averaged GF without SOI, see (1.9), and

$$\delta\hat{H} = a(\sigma_1\hat{p}_y - \sigma_2\hat{p}_x) = apM, \quad M = \frac{1}{p}(p_y\sigma_1 - p_x\sigma_2), \quad M^2 = \sigma_0.$$

and we've used (6.22). From (6.23) we see that (6.25) is just the same as previously calculated GF for $b = 0$. In a completely analogous manner we take the (more general) Hamiltonian (6.2), $\Delta/2$ [defined in (6.16)] instead of ap, and M from (6.3), and directly obtain an expression for the averaged GF, which will be valid *for arbitrary SOI amplitudes a and b* because I considered an infinite perturbation series expansion. O.k., may be not arbitrary, but at least it will be still valid for $x_{a,b} \sim 1$. So may be it is time to stop abusing computers and try calculate everything in an analytical way.

Differently from what I thought before the resulting averaged GF can be obtained from the GF for the system without impurities by substituting $\varepsilon = +0 \to 1/(2\tau)$. It would be interesting to check if this GF indeed solves the SCBA, which, however, appears difficult, since $\Im G_R$ is given by a long expression, which can not be easily integrated by $\vec{p}$. (The difficulty of this calculation is comparable with the conductivity calculation; so let us just believe that our GF solves SCBA.)

6.6 The chiral basis

At some point I thought that it will be cool to calculate everything in the chiral basis. This idea appeared to be deadlock (тупиковая, Sackgasseidee), but still interesting: Let us make perturbation theory in the Dresselhaus part of the SOI in (6.2):

$$\hat{H}_0 \stackrel{\text{df}}{=} U\left(\hat{H}\Big|_{b=0}\right)U^\dagger = \frac{\hbar^2\hat{p}^2}{2m} - ap\sigma_3, \quad U = U^\dagger = \clubsuit\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\frac{p_-}{p}e^{i\pi/4} \\ 1 & \frac{p_-}{p}e^{i\pi/4}\end{pmatrix}, \quad \delta\hat{H} = -b\sigma_3\frac{p_x^2-p_y^2}{p} - 2b\sigma_2\frac{p_xp_y}{p}.$$

Consequently, in the spirit of sec. 6.5 we obtain

$$G_{R/A} = \left[\left(E_F - \frac{p^2}{2m} \pm \frac{i}{2\tau}\right)\sigma_0 + \alpha p\sigma_3\right]^{-1} = \begin{pmatrix} \left[E_F - \frac{p^2}{2m} + \alpha p \pm \frac{i}{2\tau}\right]^{-1} & 0 \\ 0 & \left[E_F - \frac{p^2}{2m} - \alpha p \pm \frac{i}{2\tau}\right]^{-1}\end{pmatrix}.$$

[24] provided that $\sigma(\omega = (E_F\tau^2)^{-1}) \approx \sigma(\omega = 0)$, which is equivalent to say that $\sigma(\omega = 0)$ is a smooth (гладкая) function with finite derivatives in $\omega = 0$. For a metal I estimate $(E_F\tau^2)^{-1} \sim 17$K.

Let us now diagonalize the complete Hamiltonian:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -2\frac{p_x(a+b)-ip_y(a-b)}{\Delta_{\vec{p}}}e^{i\pi/4} \\ 1 & 2\frac{p_x(a+b)-ip_y(a-b)}{\Delta_{\vec{p}}}e^{i\pi/4} \end{pmatrix} \approx \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -e^{i\alpha} \\ 1 & e^{i\alpha} \end{pmatrix}, \quad \tan\left(\alpha - \frac{\pi}{4}\right) = \frac{a-b}{a+b}\frac{p_y}{p_x} = \sqrt{\frac{1-\delta}{1+\delta}}\tan\varphi,$$

$$G_{\mathrm{R/A}} = \begin{pmatrix} -\xi + \frac{\Delta}{2} \pm \frac{i}{2\tau} & 0 \\ 0 & -\xi - \frac{\Delta}{2} \pm \frac{i}{2\tau} \end{pmatrix}^{-1}, \quad \frac{\partial G_{\mathrm{R/A}}}{\partial\delta}\Big|_{\delta=0} = -\frac{\sigma_3}{4}\Delta_0 G^2_{\mathrm{R/A}}\cos 2\varphi.$$

Working in the chiral basis has aparent advantages:

- the "physical interpretation" of the SOI (as an effective MF) is now very visible. [Especially in the 2nd loop, where we can approximate $p \approx p_{\mathrm{F}}$ in the SOI-part of the Hamiltonian, because we are not interested in the corrections $\propto (p_{\mathrm{F}}l)^{-1}$, which, e.g., lead to corrections to the velocity operator, see (6.7).]
- We see that in case $\delta = 0$ the system behaves as a 2D metal without SOI subjected to a constant external in-plane MF, which clearly can not affect conductivity tensor (even retaining its isotropic part). This fits Sec. 8.2 very well.
- I claim even more: we can eliminate the momentum-independent part of this external MF, which should simplify the calculation.
- Some MEs of the diffuson gain mass due to the MF, but, differently from the usual bases, the characteristic convergence scale of $\int \frac{d^2k}{(2\pi)^2}\int \frac{d^2q}{(2\pi)^2}$ is $q^* \sim (x_a^2 + x_b^2)^{1/4}/l$. In the spinor basis from eq. ([73]4.5), the diffuson is diagonal even for $q \neq 0$!

However, all this fancy stuff does not work (except, may be, for the ZLA), since our simple disorder potential in the original system corresponds to a very complicated one in the chiral basis and vice versa.☹

6.7 Cooperon and diffuson

BN said that this [i.e., the employment of (13.37)] corresponds to the division of a diffuson into singlet D^{00} and triplet (D^{11}, D^{22}, D^{33}). I think, BN was wrong, see sec. 14.2 together with the footnote **??** on p. **??**. Look on eq. ([73]4.5): I could also work in the spinor basis $(\sigma_0, \sigma_+, \sigma_-, \sigma_3)$ instead of $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$. It is interesting, what would it change.

Note also that $D^{\alpha\alpha}$ can be interpreted as elements of spin DM, see p. [10]19.

STOP From cond-mat/9905028 I've got an impression that only cooperon gains mass due to the dephasing (I mean the term $1/\tau_\varphi$ in the denominator due to the interaction and ...), so that all components of a cooperon are in reality finite, while some components of the diffuson have a true pole for $q = 0$ and $\omega = 0$.

Note the symmetries:

$$\begin{aligned} X_D^{\alpha\beta}(\vec{q},\omega) &= X_D^{\alpha\beta *}(-\vec{q},-\omega), \quad X^{\alpha\beta}(\vec{q}) = X^{\beta\alpha}(\vec{q}), \quad \alpha,\beta \neq 2 \text{ or } \alpha = \beta = 2, \\ X^{\alpha\beta}(\vec{q}) &= -X^{\beta\alpha}(\vec{q}), \quad \alpha = 2, \beta \neq 2 \text{ or } \alpha \neq 2, \beta = 2, \end{aligned} \tag{6.26}$$

Concerning the symmetries of $D(\vec{q})$, see (8.17),8.18,8.19.

Finally, let us see the correspondence between my expression (**??**) for the cooperon with the cooperon components in the notation of [35], namely in ([35]6,7,13). From the comparison of my expressions for the WL correction (7.21) with ([35]6) we obtain, that

$$C^{\alpha\lambda}_{\beta\mu} = \sum_{\gamma,\gamma'=0}^{3} \sigma^{\gamma}_{\alpha\beta}\sigma^{\gamma'}_{\mu\lambda}C^{\gamma\gamma'}, \quad C^{\gamma\gamma'} = \frac{1}{4}\sum_{\alpha\beta\mu\lambda=1}^{2} C^{\alpha\lambda}_{\beta\mu}\sigma^{\gamma}_{\beta\alpha}\sigma^{\gamma'}_{\lambda\mu}. \tag{6.27}$$

6.8 Diffuson for $|a| \neq |b|$

We see that only the $(1,1)$-minor (i.e., matrix block) is affected by the SOI. So let us work with this minor in X_D and D. The following property has been checked[25] up to the precision of $(x_a, x_b, lq)^6$:

$$X_D(\vec{q}, x_a, x_b) = C_{xy}\left[X_D(\vec{q}, x_a, -x_b)\right] = C_{xy}\left[X_D(-\vec{q}, -x_a, x_b)\right] = X_D(-\vec{q}, -x_a, -x_b), \tag{6.28}$$

[25] BTW, (6.28) holds also for the diffuson (6.37).

where C_{xy} is defined in (8.17). Obviously the same relations hold[26] also for the diffuson:

$$C_{xy}[D(Q_x,Q_y,\delta=0)]=D(Q_y,Q_x,\delta=0). \tag{6.29}$$

The diffuson[27] is obtained using (**??**) with X_D given by

$$X_D^{22}=X_D^{00}-X_D^{11}+X_D^{33},\quad \frac{E_3-X_D}{x^2}\approx Y^{(0)}-\left[x^2Y^{(0,2)}+\delta Y^{(1,0)}+\delta\cdot x^2Y^{(1,2)}\right],\quad \vec{Q}\stackrel{\rm df}{=}l\vec{q}/x,\quad Q\lesssim 1, \tag{6.30}$$

$$Y^{(0)}=\frac{Q^2}{2}E_3+\frac{1}{2}\begin{pmatrix}1&0&-2iQ_x\\0&1&-2iQ_y\\2iQ_x&2iQ_y&2\end{pmatrix},\quad Y^{(1,0)}=\frac{1}{2}\begin{pmatrix}-1&0&iQ_x\\0&1&-iQ_y\\-iQ_x&iQ_y&0\end{pmatrix}, \tag{6.31}$$

$$Y^{(0,2)}=3E_3\left[\frac{Q^2}{4}+\frac{Q^4}{8}\right]+\frac{1}{2}\begin{pmatrix}1+3Q_x^2&3Q_xQ_y&-iQ_x(4+3Q^2)\\3Q_xQ_y&1+3Q_y^2&-iQ_y(4+3Q^2)\\iQ_x(4+3Q^2)&iQ_y(4+3Q^2)&2+\frac{9}{2}Q^2\end{pmatrix}, \tag{6.32}$$

$$Y^{(1,2)}=\frac{1}{4}\begin{pmatrix}3(1+2Q_x^2+Q^2)&0&-iQ_x(8+3Q^2)\\0&-3(1+2Q_y^2+Q^2)&iQ_y(8+3Q^2)\\iQ_x(8+3Q^2)&-iQ_y(8+3Q^2)&6(Q_x^2-Q_y^2)\end{pmatrix}, \tag{6.33}$$

$$8\det Y^{(0)}=2+Q^2+Q^6=(Q^2+1)(Q^4-Q^2+2)=(Q^2+1)\left(Q^2-\frac{1-i\sqrt{7}}{2}\right)\left(Q^2-\frac{1+i\sqrt{7}}{2}\right). \tag{6.34}$$

The above expressions for Y coincide with the exact result (6.36) for $Q=0$, and, apparently, with ([73]4.8,4.9). The limit SOI$\to 0$ corresponds to $Y^{(0)}=\frac{Q^2}{2}E_3$, $Y^{(0,2)}_Q=-\frac{3}{8}Q^4E_3$, and $Y^{(1,2)}=0$. The matrix $(E_3-X_D)/x^2-Q^2E_3/2+3Q^4x^2E_3/8$ is Hermitian and its $(3,3)$-element is a sum of $(1,1)$th and $(2,2)$th.

The contribution of terms from $[\dots]$ in (6.30) to the diffuson (**??**) can be taken into account perturbatively:

$$\begin{aligned}
&4\pi\nu\tau x^2D(\vec{Q})=x^2\left(E_3-X_D\right)^{-1}\approx\left\{Y^{(0)}-\left[x^2Y^{(0,2)}+\delta Y^{(1,0)}+\delta\cdot x^2Y^{(1,2)}\right]\right\}^{-1}\approx d^{(0,0)}+x^2d^{(0,2)}+\delta\cdot d^{(1,0)}+x^2\delta\cdot d^{(1,2)},\\
&d^{(0,0)}=\left[Y^{(0)}\right]^{-1},\quad d^{(0,2)}=d^{(0,0)}Y^{(0,2)}d^{(0,0)},\quad d^{(1,0)}=d^{(0,0)}Y^{(1,0)}d^{(0,0)},\\
&d^{(1,2)}=d^{(0,0)}\left\{Y^{(1,2)}+Y^{(0,2)}d^{(0,0)}Y^{(1,0)}+Y^{(1,0)}d^{(0,0)}Y^{(0,2)}\right\}d^{(0,0)}=d^{(0,0)}Y^{(1,2)}d^{(0,0)}+d^{(0,2)}Y^{(1,0)}d^{(0,0)}+d^{(0,0)}Y^{(1,0)}d^{(0,2)}=\\
&=d^{(0,0)}Y^{(1,2)}d^{(0,0)}+d^{(1,0)}Y^{(0,0)}d^{(0,0)}+d^{(0,0)}Y^{(0,0)}d^{(1,0)},\quad x^2,\delta\ll 8\det Y^{(0)}\sim 1,\quad 0\le Q\lesssim 1.
\end{aligned} \tag{6.35}$$

The resulting $D_{ij}(\vec{Q})$ will have denominator $\propto(\det Y^{(0)})^n$ where n is an integer number, see (6.34). The expression (6.34) for $8\det Y^{(0)}$ is independent on the direction of the small momentum $\vec{Q}$, so the same is true for the denominators of all components of diffuson. Consequently, an arbitrary diagram with two loops (e.g., rhs of fig. [20]**??**) has denominators [consisting of $(\det Y^{(0)})^\alpha$ for momenta $\vec{K}$, $\vec{Q}$, and $\vec{K}+\vec{Q}$] which are invariant with respect to two "mirror reflections": (i) $(K_x\to -K_x,Q_x\to -Q_x)$, (ii) $(K_y\to -K_y,Q_y\to -Q_y)$, and (iii) $(K_x\leftrightarrow K_y,Q_x\leftrightarrow Q_y)$.
It is interesting that the original Hamiltonian (6.1),6.2 does not possess any of these symmetries.

$$d^{(0,0)}=2\begin{pmatrix}\frac{1+Q_x^2}{1+Q^2}+Q_x^2\frac{2-Q^2}{2-Q^2+Q^4}&\frac{Q_xQ_y}{1+Q^2}+Q_xQ_y\frac{2-Q^2}{2-Q^2+Q^4}&\frac{2iQ_x}{2-Q^2+Q^4}\\ \frac{Q_xQ_y}{1+Q^2}+Q_xQ_y\frac{2-Q^2}{2-Q^2+Q^4}&\frac{1+Q_y^2}{1+Q^2}+Q_y^2\frac{2-Q^2}{2-Q^2+Q^4}&\frac{2iQ_y}{2-Q^2+Q^4}\\ -\frac{2iQ_x}{2-Q^2+Q^4}&-\frac{2iQ_y}{2-Q^2+Q^4}&\frac{1+Q^2}{2-Q^2+Q^4}\end{pmatrix},\qquad \text{for}\quad Q\gg 1\quad d^{(0,0)}\sim\frac{2}{Q^2}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

The corrections (due to $\delta\neq 0$) to $d^{(0,0)}$ maintain parity and мнимость, i.e., e.g., elements $(1,3)$ and $(2,3)$ are purely imaginary; $d^{(0,0)}_{1,2}\propto Q_xQ_y\times P(Q_x^2,Q_y^2)$ (where P is some polynomial), etc.

[26] To realize this, I can represent the $(E_3-X_D)^{-1}$ as $[(1+Q^2)E_3/2+\xi]^{-1}$, which is an infinite series in ξ. Since (6.29) holds for ξ, it holds for every term in this series, so it is valid also for the entire series, i.e., for the diffuson. (note that this series is really infinite, unlike the expansion (6.35) which is cut on the second term.) Or I could make a little more effort, checking this explicitly for both terms in (6.35).

[27] Note that, in general, X_D is not the same in σ- and σ-representations. There is an exception – the special case $q=0$ and $x_ax_b=0$ due to its diagonality and $X_D^{11}=X_D^{22}$. Based on our (no more present here) calculations we suspect that it always holds for $q=0$ and $x_ax_b=0$. Note (see `ORIG-BASIS/diffuson.max`) that in the original basis diffuson is non-diagonal even for $q=0$ if $x_a\neq 0$ and $x_b\neq 0$: elements 11, 23, 22, 32, 33, 44 are non-zero. Other MEs are zero because of spin summation and angular integration (i.e., without the help of imprecise $\int_{-E_F}^{\infty}d\xi\approx\int_{-\infty}^{\infty}d\xi$).

6.9 $D(q=0)$ can be calculated without expanding in SOI amplitudes

Let us introduce

$$\overline{M}=\frac{1}{\tilde{p}}(\beta p_y\sigma_1+p_x\sigma_2),\quad \overline{M}^2=\sigma_0,\quad (\sigma_0\pm\overline{M})^2=2(\sigma_0\pm\overline{M}),\quad \overline{G_{\mathrm{R/A}}}\stackrel{\mathrm{df}}{=}\frac{1}{2}\left[(\sigma_0+\overline{M})g_{\mathrm{r}}^{-}+(\sigma_0-\overline{M})g_{\mathrm{r}}^{+}\right].$$

Then

$$\overline{G_{\mathrm{R}}}(\vec{p})+\overline{G_{\mathrm{R}}}(-\vec{p})-G_{\mathrm{R}}(-\vec{p})=G_{\mathrm{R}}(\vec{p}),$$

which leads to the fact that $X_D^{22}=X_D^{00}-X_D^{11}+X_D^{33}$ (see above). For $q=0$, X_D is a diagonal matrix with elements:

$$X_D^{00}=1,\quad X_D^{11}=\frac{1+K}{1+(x_a+x_b)^2+K},\quad X_D^{33}=\frac{1}{K},\quad X_D^{22}=X_D^{00}-X_D^{11}+X_D^{33}, \tag{6.36}$$

$$K=\sqrt{[1+(x_a+x_b)^2][1+(x_a-x_b)^2]},\quad 4\pi\nu\tau D\Big|_{q=0}=\mathrm{diag}\left(\infty,1+\frac{1+K}{(x_a+x_b)^2},1+\frac{1+K}{(x_a-x_b)^2},\frac{K}{K-1}\right). \tag{6.37}$$

For $q\neq 0$:

$$X_D=\frac{1}{4\pi\nu\tau}\begin{pmatrix}\mathrm{Sp}[G_{\mathrm{R}}(\vec{p})G_{\mathrm{A}}] & 0 & 0 & 0\\ & \mathrm{Sp}[\overline{G_{\mathrm{R}}}(\vec{p})G_{\mathrm{A}}] & \mathrm{Sp}[i\sigma_3\overline{G_{\mathrm{R}}}(-\vec{p})G_{\mathrm{A}}] & \mathrm{Sp}[-i\sigma_2G_{\mathrm{R}}(-\vec{p})G_{\mathrm{A}}]\\ & & \mathrm{Sp}[\overline{G_{\mathrm{R}}}(-\vec{p})G_{\mathrm{A}}] & \mathrm{Sp}[i\sigma_1G_{\mathrm{R}}(-\vec{p})G_{\mathrm{A}}]\\ & & & \mathrm{Sp}[G_{\mathrm{R}}(-\vec{p})G_{\mathrm{A}}]\end{pmatrix}=$$

$$=\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\begin{pmatrix}g_{\mathrm{a}}^{+}g_{\mathrm{r}}^{+}+g_{\mathrm{a}}^{-}g_{\mathrm{r}}^{-} & 0 & 0 & 0\\ & \mathrm{Sp}[\overline{G_{\mathrm{R}}}(\vec{p})G_{\mathrm{A}}] & 0 & \mathrm{Sp}[-i\sigma_2G_{\mathrm{R}}(-\vec{p})G_{\mathrm{A}}]\\ & & \mathrm{Sp}[\overline{G_{\mathrm{R}}}(-\vec{p})G_{\mathrm{A}}] & \mathrm{Sp}[i\sigma_1G_{\mathrm{R}}(-\vec{p})G_{\mathrm{A}}]\\ & & & \mathrm{Sp}[G_{\mathrm{R}}(-\vec{p})G_{\mathrm{A}}]\end{pmatrix},$$

where $G_{\mathrm{A}}\equiv G_{\mathrm{A}}(\vec{p}-\vec{q})$. We can calculate X_D out of the diffusion approximation [i.e., for not small (x_a,x_b)], though the resulting expressions will be complicated.

Chapter 7

The spin-Hall effect

This section resulted in our paper [74]. Related papers: cond-mat/0706.4273, [24, 75, 67, 76, 77, 78, 70], 0405065, cond-mat/0410295, 0502478, 0503616, 0504035, 0504147, 0503616, 0505131, 0504218[1], 0506189, 0507007, 0507149, 0510114, 0509702, 0512054, 0509678, 0512458, 0601315, 0601525, 0605687, and many others.[2]

In the current vertex of the left diagram in fig. 7.1, I approximate $\vec{p} \approx p_F \vec{n}$. The relative accuracy of this approximation[3] is $\max[(E_F\tau)^{-1}, \Delta/E_F]$, so that I can think that the result of calculation is in reality multiplied by $\{1 + O(\max[(E_F\tau)^{-1}, \Delta/E_F])\}$.

Can I ignore $\vec{A}$ given by (6.8) in the current operator because it has amplitude $(E_F\tau)^{-1} \ll 1$ with respect to the main part? Yes, I could, if the main part of $\hat{\vec{j}}$ would give non-zero result. However, it gives exactly zero in case of the left diagram[4] in fig. 7.1, and gives only a correction in the first order in $(p_F l)^{-1}$ in case of the right diagram. That is why the $\propto \vec{A}$ part in the current vertex has to be taken into account in both diagrams in fig. 7.1. The diamagnetic term for spin current is zero due to the presence of σ_3 in the spin current vertex[5].

notes from V. Yudson (Trieste, June 2004) (i) there is such a small effect: when we switch on the vector potential, the spin-orbit part of the Hamiltonian changes... look on the paper of Aleiner and Fal'ko about this. (ii) a diffuson has a zero-mass mode which would give giant contribution if it would not be zero. May be in reality it gives important contribution represented by this great number weakened by some small factor like Δ/E_F. (iii) As for the cut-off in the ballistic limit: it is better to say that it is not frequency that has an infinitesimal imaginary part, but the $1/\tau$ which has an infinitesimal real part. In fact, when we have no disorder, $G_{R/A}$ are given by (6.17), so we can say instead that it is given by (6.23) with $1/(2\tau) = +0$. STOP – as I've checked, the cut-off for τ does not help us; it is essentially different from the frequency cut-off, that we really need.

7.1 The two diagrams

In this sec. we perform calculations in case $x_b = 0$ and in the (usual) σ-basis.
The problem was inspired by the paper [75]. What they do is just calculating the diagram with a diffuson in fig. 7.1. The method used there I could not understand. It seemed to me perverse and probably is incorrect. So we decided to do it using standart disorder averaging technique.

The spin current formula can be obtained from the usual charge current formula [20] with $\vec{A} \to 0$, where $\hat{\vec{j}}(\vec{p})$ given by (1.5) multiplied by $\sigma_3\hbar/(2e)$ according to (6.13). For G_K we use (3.38) with $\vec{n}$ substituted by $\vec{n} - ie\vec{A}/(cp_F)$ where $\vec{A}$ is given by (6.8). We imply that the electric field is applied along the x axis, while the spin current is measured along the

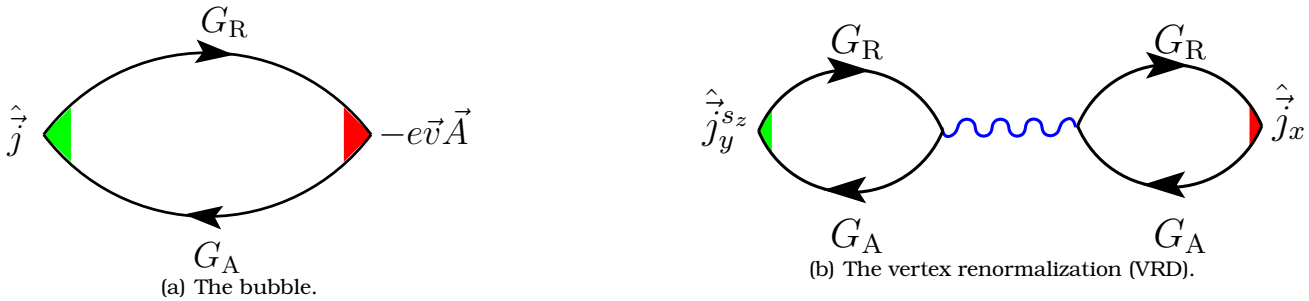
[1] 12.04.2005 DL was going to send an email to Haldane about his cond-mat/0504218.

[2] J. Sinova in Korea has called all these spin-Hall activities a "Field". May be he is right, if we judge according to the number of papers. Anyway, in 2008 this "Field" seems to be dead.

[3] See the discussion on p. 51.

[4] Using (6.3), we obtain that in the most general case (when $\alpha \neq 0$ and $|\beta| \neq 1$) the left diagram in fig. 7.1 changes sign with respect to $p_x \to -p_x$ or $p_y \to -p_y$. This means that only the part of the vertex $\propto \vec{A}$ gives non-zero contribution to the left diagram in fig. 7.1, and this does not depend on whether we approximate $|\vec{p}| \approx p_F$ or not. At first I thought this is also true for the right diagram, however, as the authors of [79] have pointed out, the $\hat{\vec{p}}$ term in the current vertex gives the correction of the same order, as the $\propto \vec{A}$ term, so it has to be taken into account, see sec. 7.3. As a result, our the first version of paper [74] was incorrect.

[5] For the normal current - see Sec. 8.1.



(a) The bubble.

(b) The vertex renormalization (VRD).

Figure 7.1: Diagrams for the spin Hall (and charge) conductivity in the zero-loop approximation. For vertices see (7.1).

y axis. Using (3.14) as the definition of a conductivity, we get the formula for the spin-Hall conductivity[6]:

$$\sigma = \frac{e}{2\pi m^2}\,\mathrm{Sp}\left[\frac{\sigma_3}{2} p_y \hat{G}_R\left(p_x - \frac{e}{\hbar c}\tilde{A}_x\right)\hat{G}_A\right]. \tag{7.1}$$

After the averaging (7.1) produces (in the zero-loop approximation) two diagrams in fig. 7.1. For both diagrams we need to calculate

$$\vec{P} = \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\vec{n} G_A(p, E-\omega)\frac{\sigma_3}{2} G_R(p,E) = \frac{\pi\nu x_a \tau/2}{x_a^2 + (1-i\omega\tau)^2}(\sigma_1, \sigma_2, 0), \tag{7.2}$$

Consider the first diagram. One can see that the term $\propto p_x$ makes zero contribution to (7.1), because

$$\mathrm{Sp}\left[\sigma_3 G_R(\vec{p}) G_A(\pm\vec{p})\right] = 0. \tag{7.3}$$

The rest of the expression gives us

$$\sigma^{(0)} = \frac{ev^2}{2\pi}\,\mathrm{Sp}\left[P_y\left(-\frac{e}{cp_F}\tilde{A}_x\right)\right],$$

and the result is

$$\sigma^{(0)}(\omega) = -\frac{e\hbar\nu}{4m}\frac{x_a^2}{x_a^2 + (1-i\omega\tau)^2} = -\frac{e}{8\pi}\frac{x_a^2}{x_a^2 + \lambda^2}, \quad \lambda = 1 - i\omega\tau, \tag{7.4}$$

where we used (13.11). Note that $\Re\sigma^{(0)}$ does not and $\Im\sigma^{(0)}$ does become large in the limit when $\tau^{-1} \ll \omega = \Delta$. Now let us calculate the second diagram in fig. 7.1 with two bubbles. One obtains that

$$p_F \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\vec{n} G_R(p,E) G_A(p, E-\omega) = 0, \quad \text{but remmember about (7.14)!} \tag{7.5}$$

$$Q = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} G_R(p,E)\sigma_2 G_A(p, E-\omega) = \frac{2\pi\nu\tau}{1-i\omega\tau} \times \frac{x_a^2/2 + (1-i\omega\tau)^2}{x_a^2 + (1-i\omega\tau)^2}\sigma_2. \tag{7.6}$$

Note that

$$X_D^{22} = \frac{1}{4\pi\nu\tau}\,\mathrm{Sp}\left[\sigma_2 Q\right], \quad X_D^{22}\Big|_{\omega=0} = \frac{1}{2}\left[1 + \frac{1}{1+x_a^2}\right] = \frac{1 + x_a^2/2}{1+x_a^2},$$

where X_D^{22} is a notation defined in[20]. From (7.3) and (7.5) it follows that for both diagrams in fig. 7.1 it is safe to approximate the spin current vertex as $p_y\frac{\sigma_3}{2} \approx p_F n_y \frac{\sigma_3}{2}$, since the corrections will give contribution to σ^z_{yx} not larger than $\sim \frac{e}{p_F l}$.

The contribution of the second diagram in fig. 7.1 equals to[7]

$$\sigma^{(1)}_{yx} = \sum_{\mu=0}^{3} \frac{ep_F}{2\pi m^2}\,\mathrm{Sp}\left[P_y\sigma_\mu\right]\frac{1}{4\pi\nu\tau}\frac{1}{1 - X_D^{\mu\mu}}\,\mathrm{Sp}\left[-p_R(Q+Q')\sigma_\mu\right], \tag{7.7}$$

[6] DL: (7.1) is the generalized Kubo formula. OS: about (generalized) Kubo formula see pp. 54-60 in Russian edition of: Madelung, "Physics of solid state. Localized states."

[7] The multiplier $\frac{1}{2}$ comes from (13.37). Before MSH has pointed out our mistake in [74] v. 1, I had Q in (7.7) instead of $(Q+Q')$, so that the result was that for $\omega = 0$ the contribution of two diagrams in fig. 7.1 was $|e|/(4\pi)$ instead of 0. (that is, $\sigma^{z(0)}_{yx} + \sigma^{z(1)}_{yx} = 0$, which is the result of MSH [79]) This mistake has been corrected in [74] v. 2.

where Q' is given by (7.14). One checks that only $\mu = 2$ gives non-zero contribution to (7.7), and

$$\sigma^{(1)}_{yx} = \sigma^{(0)}_{yx} \frac{1}{1 - X^{22}_D} \mathrm{Sp}\left[\frac{Q + Q'}{m\tau}\frac{\sigma^2}{2}\right], \quad \sigma^{z(0)}_{yx} + \sigma^{z(1)}_{yx} = \sigma^{z(0)}_{yx} \frac{1 + \mathrm{Sp}\left[\frac{Q'}{m\tau}\frac{\sigma^2}{2}\right]}{1 - \mathrm{Sp}\left[\frac{Q}{m\tau}\frac{\sigma^2}{2}\right]}. \tag{7.8}$$

From (7.8) we observe that the renormalization of the current vertex (=VRD; corresponds to considering the second diagram in fig. 7.1) results in the cancellation of the $\propto \tilde{A}_x$ term in the electric vertex. In other words, I can forget about the second diagram in fig. 7.1) if I ignore the term $\propto \tilde{A}_x$ when calculating the first diagram.

It is especially important that massless singlet diffuson mode X^{00}_D does not give contribution; otherwise $\sigma^{(1)}$ could be much larger than $\sigma^{(0)}$. Using (7.7) together with (**??**) and (7.6), we obtain (with λ is defined in (7.4))

$$\sigma^{z(1)}_{yx} = -\sigma^{z(0)}_{yx} \frac{x_a^2/2}{x_a^2 + \lambda^2} \frac{1}{\lambda - \frac{x_a^2/2 + \lambda^2}{x_a^2 + \lambda^2}}, \quad \sigma^{z(0)}_{yx} + \sigma^{z(1)}_{yx} = \sigma^{z(0)}_{yx} \frac{\lambda - 1}{\lambda - \frac{x_a^2/2 + \lambda^2}{x_a^2 + \lambda^2}}, \tag{7.9}$$

$$\sigma^{z}_{yx} = -\frac{|e|}{8\pi} \frac{i\omega\tau x_a^2}{(1 - 2i\omega\tau)\frac{x_a^2}{2} - i\omega\tau\lambda^2}, \quad \sigma^{z}_{yx}(\omega = 0) = 0, \tag{7.10}$$

which is the same as ([79]25).

7.2 The renormalization of the charge current vertex

The result (7.10) can be also obtained from calculating only the "bubble" (8.7) with one of speed operators substituted with its renormalized value. From Sec. 8.3.2 we conclude that at zero frequency

$$\text{for } \omega = 0 \quad \mathrm{Sp}\left[\tilde{\sigma}_i G^E_{\mathrm{R}} \frac{p_j}{m} G^E_{\mathrm{A}}\right] = 2m\tau \begin{pmatrix} 0 & b - a \\ a + b & 0 \end{pmatrix}, \quad i, j = 1, 2. \tag{7.11}$$

When calculating (7.11), it is essential that $p \neq p_{\mathrm{F}}$ in (6.19). Looking now into Sec. 6.7, we write

$$\mathrm{Sp}\left[\tilde{\sigma}_i G_{\mathrm{R}} \left(-\frac{e}{mc}\tilde{A}_j\right) G_{\mathrm{A}}\right] = 2\tau \begin{pmatrix} -\frac{X^{12}_D}{\lambda_1} & \frac{X^{11}_D}{\lambda_2} \\ -\frac{X^{22}_D}{\lambda_1} & \frac{X^{21}_D}{\lambda_2} \end{pmatrix} = 2m\tau \begin{pmatrix} 0 & (a - b)X^{11}_D \\ -(a + b)X^{22}_D & 0 \end{pmatrix}, \tag{7.12}$$

where MEs X_D are given by (6.36). Finally, we take the sum of (7.11) and (7.12), and see that taking vertex renormalization into account results in the cancellation of the anomalous contribution to the velocity:

$$\text{for } \omega = 0 \quad \hat{\tilde{v}} = \hat{v} + \sum_{\gamma,\gamma'=0}^{3} \tilde{\sigma}_\gamma D^{\gamma\gamma'} \mathrm{Sp}\left[\tilde{\sigma}_{\gamma'} G^E_{\mathrm{R}} \hat{v} G^E_{\mathrm{A}}\right] = \frac{\vec{p}}{m}, \tag{7.13}$$

where $D^{\gamma\gamma'}$ is given by (**??**),6.36. In our convenient $\tilde{\sigma}$-representation, the diffuson is diagonal: $D^{\gamma\gamma'} \equiv D^{\gamma\gamma'}(q = 0, \omega) = D^{\gamma\gamma}\delta_{\gamma,\gamma'}$. Note that, because of the ω-corrections to (7.11), (7.13) is not exact when $\omega \neq 0$. In case when $0 < \omega\tau \ll x^2 \ll 1$ the main contribution of the anomalous part of the velocity operator (6.8) is cancelled, so that (7.13) has corrections $\propto x_1 W^2$; in case when $0 < x^2 \ll \omega\tau \ll 1$, the main contribution of the anomalous part is not cancelled, so that (7.13) has corrections $\propto x_1 W$.

Finally, the fact that diffuson's divergence is cancelled by the bubble, means that actually the vertex renormalization correction can not be called "local" or "non-local". I mean, it is correct to say that ZLA-contribution is local, despite that it contains the vertex-renormalization diagram which has a diffuson.

7.3 The correction in the current vertex due to MSH

STOP Expand and/or rewrite this section together with sec. 3.7. I realized what is written here thanks to the private comμnication with the authors of [79]. In fact, the recipe is simple and is written in this file:

Now let us calculate STOP insert $\hbar$ from here till the end of this subsection STOP and rewrite (7.14) without approximation $\alpha p \approx \Delta/2$

$$Q'_\omega = -\frac{\sigma_2}{2m\alpha} \mathrm{Sp}\left[\sigma_2 \int \frac{\mathrm{d}^2 p}{(2\pi)^2} G^E_{\mathrm{R}}(p) p_x G^{E-\omega}_{\mathrm{A}}(p)\right] = \frac{\sigma_2}{8\pi m\alpha} \left[I_\omega(\alpha) - I_\omega(-\alpha)\right], \tag{7.14}$$

where STOP вставить частоту $\omega \neq 0$:

$$I_0(\alpha) \equiv \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p \frac{p^2}{\left(E - \alpha p - \frac{p^2}{2m} + E_\mathrm{F} + \frac{i}{2\tau}\right)\left(E - \alpha p - \frac{p^2}{2m} + E_\mathrm{F} - \frac{i}{2\tau}\right)}.$$

If I approximate $p \approx p_\mathrm{F}$ in the denominator of $I_0(\alpha)$, I get:

$$I_0(\alpha p) \approx 2\pi\tau m^{3/2}\left(z^2 + \frac{1}{4\tau^2}\right)^{\frac{1}{4}} \sqrt{1 + \frac{1}{\sqrt{1 + \frac{1}{4z^2\tau^2}}}}, \quad z = E - \frac{\Delta}{2} + E_\mathrm{F}. \tag{7.15}$$

Substituting $E = 0$ and disregarding terms $\propto 1/(E_\mathrm{F}\tau)$, we obtain

$$I_0(\Delta) - I_0(-\Delta) = -2\pi\tau m^{3/2}\sqrt{E_\mathrm{F}}\left(\sqrt{|2 + \Delta/E_\mathrm{F}|} - \sqrt{|2 - \Delta/E_\mathrm{F}|}\right), \tag{7.16}$$

which gives us in the first order in Δ/E_F an **incorrect result:**

$$I_0(\Delta) - I_0(-\Delta) \sim -\pi\tau p_\mathrm{F} m\frac{\Delta}{E_\mathrm{F}}, \quad Q' = -m\tau\sigma_2/2. \longleftarrow \text{incorrect!!!} \tag{7.17}$$

Let us now recalculate Q'_ω from (7.14) with $G_{\mathrm{R/A}}$ given by (6.23) with $\Delta/2$ substituted by $\alpha p \neq \alpha p_\mathrm{F}$. In this case instead of (7.15) we have

$$I_\omega(\pm\Delta) \equiv \int_0^\infty \mathrm{d}p \frac{p^2}{\left(E'_\mathrm{F} - \frac{(p\pm p_\mathrm{R})^2}{2m} + \frac{i}{2\tau}\right)\left(E'_\mathrm{F} - \omega - \frac{(p\pm p_\mathrm{R})^2}{2m} - \frac{i}{2\tau}\right)} = \tag{7.18}$$

$$= \int_{\pm p_\mathrm{R}}^\infty \mathrm{d}p \frac{(p \mp p_\mathrm{R})^2}{\left(E'_\mathrm{F} - \frac{p^2}{2m} + \frac{i}{2\tau}\right)\left(E'_\mathrm{F} - \omega - \frac{p^2}{2m} - \frac{i}{2\tau}\right)} = I_1^\pm \mp I_2, \quad E'_\mathrm{F} \equiv E + E_\mathrm{F} + E_\mathrm{R},$$

$$I_1^\pm \equiv \int_0^\infty \mathrm{d}p \frac{(p \mp p_\mathrm{R})^2}{\left(E'_\mathrm{F} - \frac{p^2}{2m} + \frac{i}{2\tau}\right)\left(E'_\mathrm{F} - \omega - \frac{p^2}{2m} - \frac{i}{2\tau}\right)},$$

$$I_1^+ - I_1^- = \int_0^\infty \mathrm{d}p \frac{-4pp_\mathrm{R}}{\left(E'_\mathrm{F} - \frac{p^2}{2m} + \frac{i}{2\tau}\right)\left(E'_\mathrm{F} - \omega - \frac{p^2}{2m} - \frac{i}{2\tau}\right)} \approx \frac{-8\pi m\tau p_\mathrm{R}}{1 - i\omega\tau} = \frac{-2\pi m\tau p_\mathrm{F}}{1 - i\omega\tau}\frac{\Delta}{E_\mathrm{F}},$$

$$I_2 \equiv \int_0^{p_\mathrm{R}} \mathrm{d}p \frac{(p - p_\mathrm{R})^2}{\left(E'_\mathrm{F} - \frac{p^2}{2m}\right)^2 + \frac{1}{4\tau^2}} \approx \frac{1}{3}\frac{(2mE_\mathrm{R})^{3/2}}{E_\mathrm{F}'^2 + \frac{1}{4\tau^2}} = m\tau p_\mathrm{F} \times \frac{1}{48 p_\mathrm{F} l}\left(\frac{\Delta}{E_\mathrm{F}}\right)^3,$$

where we assumed that $E_\mathrm{R} = m\alpha^2/2 \ll E_\mathrm{F}$. Note: if I substitute p^2 with p_F^2 in the numerator of (7.18), and say that in the denominator $\frac{p^2}{2m} - E_\mathrm{F} \approx v_\mathrm{F}(p - p_\mathrm{F})$, then I get twice less result, that is, (7.17). If I substitute p^2 with p_F^2 in the numerator of (7.18), but keep the denominator untouched, I get $I(\pm\Delta) \sim \pi m\tau p_\mathrm{R}(E_\mathrm{F}\tau)^{-1}$. This means that the linearizion or other tricks with the denominator, like substituting $\alpha p \to \Delta/2$ change *drastically* (7.18). Finally, the correct expression for Q' is

$$Q' = -m\tau\sigma_2/\lambda. \tag{7.19}$$

Just to save myself from paranoia, I've experimented with

$$2\pi\nu\tau = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} g_\mathrm{r}^E(\vec{p}) g_\mathrm{a}^E(\vec{p}) \propto \int_0^\infty \mathrm{d}p p g_\mathrm{r} g_\mathrm{a}. \tag{7.20}$$

If in (7.20) I approximate $p \approx p_\mathrm{F}$ in the denominator, the result is not changed; if in (7.20) I approximate $p \approx p_\mathrm{F}$ in the denominator and linearize the denominator, the result still is not changed; From here I conclude that normally these approximations must work, so that the case of (7.18) is really special.

On p. 490 of [5] we read that it is essential that in $\langle G_{\mathrm{R/A}}(\vec{p},\vec{p}\,')\propto 2\pi\delta(\vec{p}-\vec{p}\,')G_{\mathrm{R/A}}(\vec{p})\rangle$ $|\vec{p}|$ and $|\vec{p}\,'|$ are of the order of p_F.

Note that all the values of p in I_2 are far away from p_F, thus if this integral would give important contribution, this would mean that the values of p far away from p_F are important in (7.14). This would mean that we are in trouble because Landau Fermi-liquid theory is broken by the fact that excitations far away from the Fermi surface become important[8].

To conclude: at first we suspected that in $I(\Delta)$ the values of momentum far-away from the Fermi energy are relevant; however finally we realize that this does not happen. In the following subsection it is explained more in detail.

[8] In this case we would not be able to approximate $\alpha p \approx \alpha p_\mathrm{F}$ in the denominator of the Green function (6.23), and thus τ would be very different from τ, and the expression for the diffuson would be different from (6.37), Sec. 6.8, etc.

7.4 The weak localization correction to σ^z_{yx}

Let us calculate the weak localization correction to σ^z_{yx}. It is given by the diagram in fig. 3.1(b) with $q=0$ and corrected vertices. Let us use the first part[9] of (13.37).

Then we see that the diagram equals to

$$\frac{e}{2\pi m^2}\sum_{\gamma,\gamma'=0}^{3}\int\frac{\mathrm{d}^2q}{(2\pi)^2}\times C_{\gamma\gamma'}(\vec q)\,\mathrm{Sp}\left\{\int\frac{\mathrm{d}^2p}{(2\pi)^2}G^{E-\omega}_{\mathrm{A}}(\vec p)\frac{\sigma_3}{2}\hat p_y G^{E}_{\mathrm{R}}(\vec p)\sigma^{\gamma}\times\right.$$
$$\left.\times\left[\sigma^{\gamma'}G^{E}_{\mathrm{R}}(\vec q-\vec p)\left(q_x-p_x-\frac{e}{c}\tilde A_x\right)G^{E-\omega}_{\mathrm{A}}(\vec q-\vec p)\right]^T\right\}\approx$$
$$\approx\frac{e}{2\pi m^2}\sum_{\gamma\gamma'=0}^{3}\mathrm{Sp}\left\{\int\frac{\mathrm{d}^2p}{(2\pi)^2}G^{<}(\vec p)\sigma^{\gamma}\left[\sigma^{\gamma'}G^{>}(-\vec p)\right]^T\right\}\int\frac{\mathrm{d}^2q}{(2\pi)^2}C^{\gamma\gamma'}(\vec q),\tag{7.21}$$

$$G^{<}(\vec p)=G_{\mathrm{A}}(\vec p)p_y\frac{\sigma_3}{2}G_{\mathrm{R}}(\vec p),\quad G^{>}(\vec q-\vec p)=G_{\mathrm{R}}(\vec q-\vec p)\left(q_x-p_x-\frac{e}{c}\tilde A_x\right)G_{\mathrm{A}}(\vec q-\vec p),\tag{7.22}$$

where components of the cooperon $C^{\gamma\gamma'}(\vec q)$ for $q=0$ are given by (**??**). Due to the fact that I have not yet calculated $C^{\gamma\gamma'}(\vec q)$ for $q\neq 0$, I can use ([35]13) together with (6.27) in order to evaluate $\int\frac{\mathrm{d}^2q}{(2\pi)^2}C^{\gamma\gamma'}(\vec q)$. Then we note that its off-diagonal elements are zeros, and diagonal ones are (see [35] for notations)

$$\int\frac{\mathrm{d}^2q}{(2\pi)^2}C^{\gamma\gamma'}(\vec q)=\frac{2}{\pi l^2 m\tau}(f,f,\ln,f)\delta_{\gamma\gamma'}.\tag{7.23}$$

The approximation (7.23) corresponds to taking into account only diagonal terms of the 4×4 Cooperon matrix. These terms are special because only they can produce logarithms $\ln x_a$ or $\ln L_\varphi$. The same is true for the approximation (7.21) in the sence that corrections to this approximation do not contain terms logarithmic in x_a or L_φ. Note that the assumption $x^2\ll 1$ is mandatory for obtaining logarithm; if $x_a^2\gtrsim 1$, only massless cooperon element [that is, $C_{22}(\vec q)$] will give us $\ln L_\varphi/l$; the others *will not* produce $\ln x_a$.

Using this result one obtains that, in the electric vertex, $-p_x$ gives zero contribution.

$$i_1(x_a)=\nu\int_{-\infty}^{\infty}\mathrm{d}\xi g_{\mathrm{r}}^{+2}g_{\mathrm{a}}^{-2}=\frac{2\pi\nu\tau^3}{3(1-ix_a)^3},\quad i_2(x_a)=\nu\int_{-\infty}^{\infty}\mathrm{d}\xi g_{\mathrm{r}}^{+}g_{\mathrm{r}}^{-}g_{\mathrm{a}}^{-2}=-4\pi\nu\tau^3\frac{1-ix_a/2}{(1-ix_a)^2}.$$

The "undressed" Hikami box in fig. 7.2

$$\clubsuit V^{\gamma}=-\frac{1}{p_{\mathrm{R}}p_{\mathrm{F}}}\,\mathrm{Sp}\left\{\int\frac{\mathrm{d}^2p}{(2\pi)^2}G^{<}(\vec p)\sigma^{\gamma}\left[\sigma^{\gamma}G^{>}(-\vec p)\right]^T\right\}.$$

Then

$$V^0=\frac{i}{2\sqrt2}\Im i_1(x_a),\quad \Im i_1(x_a)=\operatorname*{Asym}_x i_1(x_a).$$
$$V^1=\frac{i}{16\sqrt2}\operatorname*{Asym}_{x_a}\left[-9\Im i_1(x_a)+2\Re i_2(x_a)\right],\quad V^2=0,$$
$$V^3=\frac{i}{16\sqrt2}\operatorname*{Asym}_{x_a}\left[i_1(x_a)+2\Re i_2(x_a)\right]$$

One notes (without evaluation of integrals i_{12}) that $\sum_\gamma V^\gamma=0$, so that the result is zero. The corrections to the Hikami box in fig. 7.2(a) and in fig. 7.2(b) are

$$V^{\gamma}_{12}=\frac{1}{2m\tau}\sum_{\mu=0}^{3}A^{\gamma\mu}_{12}B^{\mu\gamma}_{12}=\frac{1}{2m\tau}(A_{12}B_{12})_{\gamma\gamma},\tag{7.24}$$

[9] By using the second part of (13.37), we can rewrite (7.22) in another way:... The problem is that I don't know how to calculate $KG_{\mathrm{R/A}}K^{-1}$ taking spin into account.

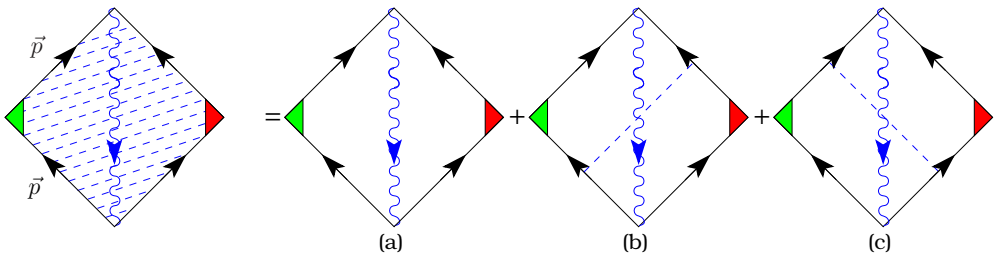


Figure 7.2: Weak localization diagrams. The spin current vertex is on the lhs (the green one).

where

$$A_1^{\gamma\mu}(\vec{q}) = \mathrm{Sp}\left\{\int \frac{\mathrm{d}^2p}{(2\pi)^2} G^<(\vec{p})\sigma^\gamma G_\mathrm{A}^T(\vec{q}-\vec{p})\sigma^\mu\right\}, \quad B_1^{\mu\gamma'}(\vec{q}) = \mathrm{Sp}\left\{\int \frac{\mathrm{d}^2p'}{(2\pi)^2}\sigma^{\gamma'} G^>(\vec{q}-\vec{p}\,')\left[G_\mathrm{A}(\vec{p}\,')\sigma^\mu\right]^T\right\}, \tag{7.25}$$
$$A_2^{\gamma'\mu}(\vec{q}) = \mathrm{Sp}\left\{\int \frac{\mathrm{d}^2p}{(2\pi)^2} G^<(\vec{p})\sigma^\mu\left[\sigma^{\gamma'} G_\mathrm{R}(\vec{q}-\vec{p})\right]^T\right\}, \quad B_2^{\mu\gamma}(\vec{q}) = \mathrm{Sp}\left\{\int \frac{\mathrm{d}^2p'}{(2\pi)^2}\sigma^\mu G_\mathrm{R}(\vec{p}\,')\sigma^\gamma\left[G^>(\vec{q}-\vec{p}\,')\right]^T\right\},$$

and

$$(A_1^{2\mu})^* = -A_2^{2\mu}, \quad (B_1^{\mu 2})^* = -B_2^{\mu 2}, \quad (A_1^{\gamma\mu})^* = A_2^{\gamma\mu}, \quad (B_1^{\mu\gamma})^* = B_2^{\mu\gamma}, \quad \gamma \neq 2, \tag{7.26}$$

so that we immediately see that $\forall\gamma V_1^\gamma + V_2^\gamma \in \mathbb{R}$ in (7.24). Mote that symmetries (7.26) are conserved by renormalizations (7.27), and remains true also for the case $\vec{q} \neq 0$, that is, $\forall\gamma V_1^\gamma(\vec{q}) + V_2^\gamma(\vec{q}) \in \mathbb{R}$.

If I approximate the electric vertex in (7.1) as $\left(p_x - \frac{e}{c}\tilde{A}_x\right) \approx \left(p_\mathrm{F} n_x - \frac{e}{c}\tilde{A}_x\right)$, then *after integration over momentum* I see that the main part of electric vertex $\propto p_\mathrm{F} n_x$ gives zero contribution. I was unable to demonstrate the fact that it gives zero, playing around with substituting $p_x \to -p_x$ in B_{12} and utilizing $G_{\mathrm{R/A}}(-p_x, p_y) = G^T_{\mathrm{R/A}}(p_x, p_y)$. Moreover, as I emphasized, I obtain this cancellation only after performing the integration. This brings me the feeling that, analogously like we saw it in the zero loop approximation with diagrams on fig. 7.1, I can not approximate the vertex in this manner. Instead, it's time to remember about the lesson of MSH from the sec. 7.3.

So, I calculated it. In both vertices (electric and spin currents) I *did not* approximate $\vec{p} \approx p_\mathrm{F}\vec{n}$. Then I saw that

- the diagram in fig. 7.2(c) is a complex conjugate of the diagram in fig. 7.2(b);
- The only non-zero elements of A in (7.25) are $(0,2),(1,3),(2,0),(3,1)$.
- The only non-zero elements of B in (7.25) are $(1,3),(3,1)$, and $B^{13} = -B^{31}$.
- Thus the result would be zero if $A^{13} = A^{31}$, which is *not true*.

As a result, $V^{00} = V^{22}$, so that the term $\propto \ln L_\varphi/L$ does not contribute and the result $\propto \ln x_a$ remains finite even when $L_\varphi \to \infty$, given that x_a is finite (though small[10]). This result is strange. Since

$$\dot{\hat{s}}_k(t) = -2m\alpha\hat{j}_k^{s_z}(t), \quad k = x, y,$$

the constant spin current would result in an infinite growth of magnetization per electron (in a non-interacting system), which is impossible. I tried hard to get zero, and I was unable to. I did not analysed the anisotropic part of diagrams in fig. 7.2(b) and 7.2(c). May be it could cancel my non-zero contribution (although, if it behaves in the same way, as for the diagram on fig. 7.2(a), it is impossible). Due to the lack of time, I have to stop here. I conclude that there is a mistake in my calculation. The mistake must be not simple, but deep.

– Due to this, at first I wanted to expel the calculation of the weak localization from our paper [74]v. 2. However, I still need weak localization diagrams if I want to achieve accuarcy $\propto e$ in the calculation of σ^z_{yx}. See [74]v. 3 and my presentation .

[10]The assumption $x_a \ll 1$ is essential for the approximation $q \to 0$ in the Hikami box. E.g., in case of the diagram in fig. 7.2(a) the anisotropic part $\propto q_x^2 - q_y^2$ of the Hikami box has a large amplitude, and we would be obliged to take its contribution into account for not small x_a. However $\propto q_x^2 - q_y^2$ must give us an additional power of x_a in the numerator after integration.

A fresh idea from DL: I have forgotten about the possibility of the renormalization of diagrams in fig. 7.2 with diffusons carrying zero momentum. [As we know from (7.13), taking them into account results in the cancellation of the term $\frac{e}{c}\tilde{A}_x$ in (7.22).]

$$\begin{aligned}\tilde{G}^<(\vec{p}) =& G^<(\vec{p}) + \sum_{\mu=0}^{3} G_A(\vec{p})\sigma^\mu G_R(\vec{p}) D^{\mu\mu}\,\mathrm{Sp}\left\{\sigma^\mu \int \frac{d^2p}{(2\pi)^2} G^<(\vec{p})\right\}\\ \tilde{G}^>(-\vec{p}) =& G^>(-\vec{p}) + \sum_{\mu=0}^{3} G_R(-\vec{p})\sigma^\mu G_A(-\vec{p}) D^{\mu\mu}\,\mathrm{Sp}\left\{\sigma^\mu \int \frac{d^2p}{(2\pi)^2} G^>(-\vec{p})\right\},\end{aligned} \tag{7.27}$$

where only terms with $\mu = 2$ are non-zero. Note that they have to obey the following property:

$$\begin{aligned}\sigma_{yx}^{z(0)} &= \frac{e}{2\pi m^2}\,\mathrm{Sp}\left[G^<(\vec{p})\left(p_x - \frac{e}{c}\tilde{A}_x\right)\right] = \frac{e}{2\pi m^2}\,\mathrm{Sp}\left[\frac{\sigma_3}{2} p_y G^>(\vec{p})\right],\\ 0 = \sigma_{yx}^{z} = \sigma_{yx}^{z(0)} + \sigma_{yx}^{z(1)} &= \frac{e}{2\pi m^2}\,\mathrm{Sp}\left[\tilde{G}^<(\vec{p})\left(p_x - \frac{e}{c}\tilde{A}_x\right)\right] = \frac{e}{2\pi m^2}\,\mathrm{Sp}\left[\frac{\sigma_3}{2} p_y \tilde{G}^>(\vec{p})\right],\end{aligned}$$

where different expressions for $\sigma_{yx}^{z(0)}$ and $\sigma_{yx}^{z(1)}$ are given in sec. 7.1. From (7.2),7.6,7.14,7.19 we get

$$\begin{aligned}&\clubsuit\,\mathrm{Sp}\left\{\sigma^2 \int \frac{d^2p}{(2\pi)^2} G^<(\vec{p})\right\} \approx \frac{\pi\nu x_a \tau p_F}{x_a^2+1}, \quad \mathrm{Sp}\left\{\sigma^2 \int \frac{d^2p}{(2\pi)^2} G^>(\pm\vec{p})\right\} = \mathrm{Sp}[-p_R Q],\\ &\clubsuit\,\mathrm{Sp}\left\{\sigma^2 \int \frac{d^2p}{(2\pi)^2} \tilde{G}^<(\vec{p})\right\} \approx \frac{\pi\nu x_a \tau p_F}{x_a^2+1}, \quad \mathrm{Sp}\left\{\sigma^2 \int \frac{d^2p}{(2\pi)^2} \tilde{G}^>(\pm\vec{p})\right\} = \mathrm{Sp}[-p_R\,(Q+Q')]\end{aligned}$$

Note that the renormalization on the rhs (i.e., of the CCV) corresponds to the cancellation of the term $\propto \tilde{A}_x$ in CCV, while the renormalization on the lhs (i.e., of the spin current vertex) is just zero (as I've checked). It must be zero, because it would lead to a term $\propto \frac{1}{p_F l}$ in the spin-Hall conductivity, which is $\neq 0$ also when $x_a = 0$. After the elimination of the term $\propto \tilde{A}_x$ in the current vertex, one can see that B_1 in (7.25) is antisymmetric. The contribution to the weak localization is proportional to

$$\propto \Re\left\{A_1^{02}B_1^{02} + A_1^{20}B_1^{20}\right\} = \Re\left[B_1^{02}\left(A_1^{02} - A_1^{20}\right)\right].$$

The problem is that $A_1^{02} \neq A_1^{20}$. The statement $B_1^{02} + B_1^{20} = 0$ is valid both with and without taking $\propto \tilde{A}_x$ term into account in $G^>$. As we know, the renormalization of the electric vertex cancels this term, so let us demonstrate $B_1^{02} + B_1^{20} = 0$, calculating $G^>$ without it:

$$\begin{aligned}&\clubsuit 2G^>\Big|_{\tilde{A}_x\to 0} = g_r^- g_a^- + g_r^+ g_a^+ + M(g_r^- g_a^- - g_r^+ g_a^+),\\ &\clubsuit B_1^{02} + B_1^{20} = \mathrm{Sp}\left\{\int \frac{d^2p}{(2\pi)^2} G^>(-\vec{p})\left[G_A^T(\vec{p}), \sigma^2\right]_-\right\} = \mathrm{Sp}\left\{\int \frac{d^2p}{(2\pi)^2} G^>(-\vec{p}) i\sigma^3 \frac{p_y}{p}\left[g_a^-(\vec{p}) - g_a^+(\vec{p})\right]\right\}.\end{aligned}$$

Very similary,

$$A_1^{02} - A_1^{20} = \mathrm{Sp}\left\{\int \frac{d^2p}{(2\pi)^2} G^<(\vec{p})\left[G_A^T(-\vec{p}), \sigma^2\right]_-\right\} = -\,\mathrm{Sp}\left\{\int \frac{d^2p}{(2\pi)^2} G^<(\vec{p}) i\sigma^3 \frac{p_y}{p}\left[g_a^-(\vec{p}) - g_a^+(\vec{p})\right]\right\} \tag{7.28}$$

One can see that it is non-zero because the vertex in $G^<(\vec{p})$ (7.22) is just $\sigma^3 p_y$.

From (7.28) we deduce that $A_1^{02} - A_1^{20} = 0$ if I would substituted the spin vertex on the left with the electric one. Thus it is correct that in [35] the renormalization of electric vertices with a diffuson is ignored together with the $\propto \tilde{A}_x$ in the current vertex. The contribution of this term to the *electric* WL is not just small (as it is written in [35]), but is *exacty* zero.

7.5 Spin current and spin precession

STOP To be reinspected, may be shortened.

We have seen that in perturbation theory the spin-Hall conductivity vanishes in the zero frequency limit, in leading

($\propto (1/p_F l)^0$) *and* subleading ($\propto (1/p_F l)^1$) order. This result suggests that the vanishing of the spin current is an exact property of the system under consideration. Indeed, we give now a simple argument to support this claim. In this section we argue based on general Hamiltonian (6.1) with *arbitrary strong* both Rashba and Dresselhaus SOI. From the Heisenberg equation of motion for the spin $\hat{\vec{s}} = \hat{\vec{\sigma}}/2$ of the electron, i.e. $\dot{\hat{\vec{s}}}(t) \equiv \frac{d}{dt}\hat{\vec{s}}(t) = i[\hat{H}', \hat{\vec{s}}](t)$, we obtain a simple relation between spin precession and spin current STOP note that (7.29) works only in case of qadratic spectrum of electrons, see this paper!

$$-\frac{1}{2m}\dot{\hat{s}}_x(t) = a\hat{j}^{s_z}_x(t) + b\hat{j}^{s_z}_y(t), \quad -\frac{1}{2m}\dot{\hat{s}}_y(t) = a\hat{j}^{s_z}_y(t) + b\hat{j}^{s_z}_x(t), \tag{7.29}$$

which is valid unless we there are no σ_{12}-dependent terms in the Hamiltonian, other than SOI in (6.1). That is, (6.1) remains valid also in case of perpendicularly applied magnetic field $B = B_z$, external (applied) electric field, electro-electron interaction and long-range-scattering impurities.

A constant electric field, applied to a piece of a metal, or a semiconductor usually drives it into a steady state[11]. Then using (7.29) we get in case $a \neq b$

$$\dot{\vec{s}} = 0 \Longrightarrow \vec{j}^{s_z} = 0 \quad \text{if } a \neq b.$$

That is, it is clear that spin-Hall conductivity must die at $\omega = 0$. This is beatifully explained in more words in sec. [74]VII.

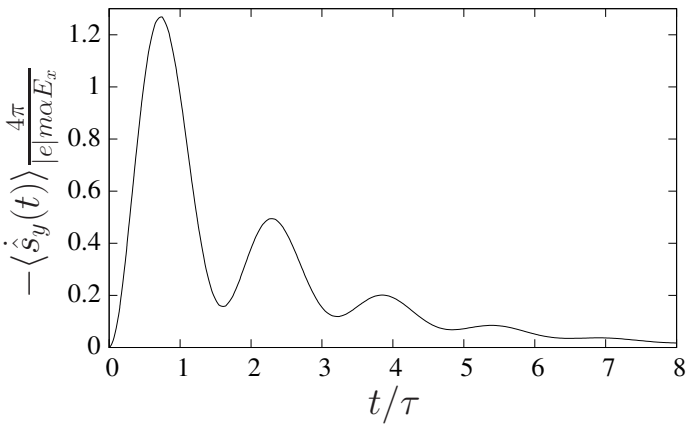


Figure 7.3: The time evolution of $\langle\dot{\hat{s}}_y(t)\rangle$ for $x_a = 4$. The period of oscillation is $1/\Delta$, and the exponential decay time is τ. Note that for any x_a, $\langle\dot{\hat{s}}_y(t)\rangle = 0$ at $t = 0$.

The approach of $\langle\dot{\hat{\vec{s}}}\rangle(t)$ to its stationary value (in the linear response regime) can be easily illustrated by taking the inverse Laplace transform of eq. (7.10) (restricting ourselves to the Boltzmann value). This involves solving a cubic equation for obtaining the poles with respect to ω. The resulting expression for $\langle\dot{\hat{s}}_y\rangle(t)$ consists of two parts (too lengthy to be written down here), one coming from the real pole and an oscillatory one coming from the pair of complex conjugate poles. For $x^2 \ll 1$ and $t \gg \tau$, only the first part is relevant, which has no oscillations and decays exponentially[12] to zero:

$$\langle\dot{\hat{s}}_y\rangle(t) = -2m\alpha\langle\hat{j}^{s_z}_y\rangle(t) = -\frac{|e|}{4\pi}m\alpha x_a^2 E_x\, e^{-t/T}, \quad t \gg \tau, \tag{7.30}$$

where $T/2 = (\Delta^2\tau)^{-1}$ is the well-known Dyakonov-Perel spin relaxation time [80], cond-mat/0601105. For $x^2 \gtrsim 1$ and $t \lesssim \tau$, the oscillatory part in $\langle\dot{\hat{s}}_y\rangle(t)$ becomes dominant, with period $1/\Delta$ and exponential decay with rate $1/\tau$; its time dependence for a particular value of x_a is illustrated in Fig. 7.3.

The above consideration can be generalized to the case with electron-electron interaction. In this case, the total spin of the system, $\sum_i \hat{s}_{i,k}$, and the total spin current, $\sum_i \hat{j}^{s_z}_{i,k}$, $k = x, y$, obey the same equation as before, i.e. eq. (7.30). Thus, the same argument goes through as well, showing that the spin current must vanish in the stationary limit also for interacting systems–provided this limit exists, which, again, we expect to be the case in the presence of any finite amount of disorder. Finally, in the absence of disorder, no stationary limit of $\langle\dot{\hat{\vec{s}}}\rangle(t)$ can be reached, i.e. the magnetization can change indefinitely, and thus there can be a finite spin current in this very particular case. [We note that the physically observable quantity, the magnetization change, vanishes for vanishing Rashba coupling.] However, when the spin current approaches a non-zero but constant value for $t \to \infty$, as obtained in the linear response regime for a clean system [0309475], the magnetization $\langle\hat{s}_k\rangle(t)$ actually grows linearly in time in the asymptotic regime $t \to \infty$. This, of course, shows a breakdown of the linear response approximation in this case since the bound for a spin 1/2, i.e. $|\langle\hat{s}_k\rangle(t)| \leq 1/2$, is violated. The terms beyond linear response would restore the bounded and oscillatory behavior of $\langle\hat{s}_k\rangle(t)$.

STOP There was another issue in Korea: without interaction, does this general argument works only for the net spin current, or for its density? EM and I colcluded that it does not work for the density, but I remember that DL said that it does. To understand it, one needs to define current density operator according to p. [81]358.

[11] What are the conditions for this? Homogeneity? Size of the system? Why MSH get $j^{s_z}_y \neq 0$ near the contacts? Why my electronic watch is not in a steady state despite its battery has constant voltage?

Chapter 8

Charge conductivity

The results of the calculation are presented in articles [66, 20] as well as in my presentation and poster. Here we calculate the anisotropy of the charge conductivity due to the Rashba and Dresselhaus SOI. The calculations are done partially on computer, see `diagrams.max/EXAMPLES/conductivity-SOI/` The anisotropy of the conductivity is connected with the anisotropy of the energy spectrum. It is also anisotropic without Dresselhaus SOI, but in the presence of MF [82]. The anisotropy may even arise in case of an isotropic energy spectrum, but in (macroscopically anisotropic) quasi-1D case. Our final result contradicts the result of the kinetic equation approach [cond-mat/0611328], thus questioning it. Other occasions, when kinetic equation can fail: cond-mat/0201007.
About Drude conductivity of the disorder-free finite-size samples, see arXiv/0801.0592.
About anomalous Hall effect, see arXiv/0804.4181.

8.1 General relations for calculating charge current

Here are the results of Sections 3.5, 3.3, 9.2, put together. Promoting WF $\psi(x) \equiv \psi(\vec{r},t)$ in (13.8) to annihilation operator (spinor) $\hat{\psi}(x) \equiv \hat{\psi}(\vec{r},t)$, we get expression for the current density operator in the SQ $\hat{\vec{j}}(\vec{r},t)$. We then split it into "normal" and "diamagnetic" contributions:

$$\hat{\vec{j}}_{SN}(\vec{r},t) = \frac{ie\hbar}{2m}\left[(\nabla\hat{\psi}^\dagger(\vec{r},t))\hat{\psi}(\vec{r},t) - \hat{\psi}^\dagger(\vec{r},t)(\nabla\hat{\psi}(\vec{r},t))\right] - \frac{e^2}{mc}\sum_{s,s'=1}^{2}\hat{\psi}^\dagger_s(\vec{r},t)\left(\vec{A}_0\delta_{s,s'} + \vec{\hat{A}}_{s,s'}\right)\hat{\psi}_{s'}(\vec{r},t), \tag{8.1}$$

$$\hat{\vec{j}}_{SD}(\vec{r},t) = \frac{e^2}{mc}\vec{A}_\omega\hat{\psi}^\dagger(\vec{r},t)\hat{\psi}(\vec{r},t), \quad \hat{\vec{j}}_S(\vec{r},t) = \hat{\vec{j}}_{SN} + \hat{\vec{j}}_{SD}, \tag{8.2}$$

where we ignored the (last) **rot**-term in (13.8). To get the current, on has to "average" $\hat{\vec{j}}(\vec{r},t)$ from (8.2) with respect to the SQ-DM of the system [cf. (13.21), 13.20]:

$$\vec{j}(\vec{r},t) = \langle\hat{\vec{j}}_S(\vec{r},t)\rangle \equiv \operatorname*{Sp}_{\text{SQ}}\left[\hat{\rho}_S(t)\hat{\vec{j}}_S(\vec{r},t)\right] \equiv \sum_{\binom{n_1,n_2\ldots\geq 0}{n_1',n_2'\ldots\geq 0}}\langle n_1,n_2\ldots|\hat{\rho}_S(t)|\ldots n_2',n_1'\rangle\langle n_1',n_2'\ldots|\hat{\vec{j}}_S(\vec{r},t)|\ldots n_2,n_1\rangle, \tag{8.3}$$

where the (perturbed by external electric field) DM depends on time, so that averaging $\langle\ldots\rangle$ is a *time-dependent* operation. Eq. (8.3) requires the knowledge of the (non-equilibrium) full SQ-DM, which we don't know how (and don't need) to calculate.[1] In order to deal with the OP-DM, we rewrite (8.3) in the FQ:

$$\vec{j}(\vec{r},t) = \operatorname*{Sp}_{\text{FQ}}\left[\hat{\rho}_1(t)\hat{\vec{j}}_F(\vec{r})\right], \quad \hat{\vec{j}}_F \equiv \hat{\vec{j}}, \quad \operatorname*{Sp}_{\text{FQ}} \equiv \operatorname{Sp}. \tag{8.4}$$

For the rest of the derivation, see [20]. Note that the derivation ideologically contradicts to arXiv/0803.1226v1; For the experimental evidence of invalidity of the free-electron-gas approximation far away from the Fermi level, see arXiv/0803.1230.

[1] In fact, we can not write the OP-DM in the SQ; see notes after (13.21).

8.2 Symmetries

There are only two notes left in addition to the appendices of [66]:

- The matrix $U \in SU(2)$ from ([66]A2) is isomorphic to the matrix $C = R^z_{-\pi/2} R^y_{\pi} \in O(3)$ from the same Eq.

$$U = u_{-\pi/2,3} u_{\pi,2} = \begin{pmatrix} 0 & \sqrt{i} \\ -\sqrt{-i} & 0 \end{pmatrix}, \quad u_{\varphi,j} = \sigma_0 \cos\frac{\varphi}{2} + i\sigma_j \sin\frac{\varphi}{2}. \tag{8.5}$$

- At first I used $[x_s = x_a + x_b, \beta = (x_a - x_b)/(x_a + x_b)]$ as the expansion parameters, however, the resulting series is not uniform. In this way I was obtaining divergences when calculating $\delta\sigma$ with the help of first non-vanishing terms in diffusons and HBs. These divergences were comming from $(0,0)$-components of diffusons. How: first I introduced dimensionless momenta like $\vec{Q} = l\vec{q}/x_s$, then x_s disappeared from the expression, so that it remained dependent only on β. And these terms diverged. I made a little investigation, why. The conclusion was: it depends how I expand $x_s^2 D^{33}(Q=0)$ in series in (x_a, x_b), i.e. it makes difference if I (i) at first expand it in x_a and then in x_b, or (ii) first expand in x_b and then in x_a.

We expand the conductivity tensor in series [66]:

$$\sigma_{xx} - \sigma_{yy} = 2\frac{e^2}{h}\frac{1}{p_F l}\sum_{m,n\geq 0} S_{mn} x^m \delta^{2n+1}. \tag{8.6}$$

From [66] we conclude that for $\sigma(|\delta| = 1) = \sigma(x = 0)$, so, if (8.6) would be correct for $|\delta| = 1$, we would stop right here and say that $\forall x\ \sigma_{xx} - \sigma_{yy} = 0$. However, the calculation [20] shows that $S_{00} \neq 0$.

8.3 ZLA and WL

8.3.1 Perturbation theory in SOI

In the ZLA we have to consider the usual "bubble" diagrams depicted on fig. 7.1(a):

$$\tilde{\sigma}^{(0)}_{\alpha\beta} = \frac{e^2}{2\pi\hbar}\mathrm{Sp}\left[\hat{v}_\alpha \hat{G}_R \hat{v}_\beta \hat{G}_A\right] = -\frac{1}{2\pi\hbar}\mathrm{Sp}\left[\tilde{j}_\alpha \hat{G}_R \tilde{j}_\beta \hat{G}_A\right], \quad \tilde{j}_\alpha = -ie\hat{v}_\alpha, G_{R/A} \to G_{R/A}\sum_{n=0}^{2}\left(\delta\hat{H} G_{R/A}\right)^n, \quad \delta\hat{H} = \frac{p_y\tilde{\sigma}_1}{m\lambda_2} = -2bp_y\tilde{\sigma}_1 = -\delta_\beta\tilde{\sigma}_1\frac{p_y}{p_F}. \tag{8.7}$$

Normally the velocity operator in diagrams obtained from (8.7) must be renormalized according to (7.13). The exception is ZLA bubble, where only one of two velocity operators must be renormalizaed. Such bubble will correspond to the sum of two diagrams 7.1.

Eq. (**??**) was the first to use in order to solve the problem. Later I've realized that (i) one can consider both Rashba and Dresselhaus SOI terms as a perturbation [BTW this noticably reduces calculation time!] and (ii) calculate in the original (not turned) basis[2]. These are ways to check the calculation; I used them, and they lead to one and the same result.

Here are the results for the SOI-dependent part of the charge conductivity (ZLA):[3]

$$\begin{aligned} \text{the Drude contribution} &= \frac{e^2}{h}\frac{1}{4E_F\tau}\begin{pmatrix}1&0\\0&1\end{pmatrix}\left[x_b^2 + x_a^2 + x_a^2x_b^2 - \frac{x_a^4 + x_b^4}{2} + O(x^6)\right], \\ \text{the vertex correction} &= -\frac{e^2}{h}\frac{1}{4E_F\tau}\begin{pmatrix}1&0\\0&1\end{pmatrix}\left[x_a^2x_b^2 - \frac{x_a^4 + x_b^4}{2} + O(x^6)\right], \end{aligned} \tag{8.8}$$

The two ZLA diagrams can be written as one bubble if we substitute one (of two) velocity operator with its renormalized value

$$\frac{\delta\sigma}{2\sigma_D} = \frac{x_a^2 + x_b^2}{8(E_F\tau)^2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{O(x^{12})}{8(E_F\tau)^2} = 2\frac{a^2 + b^2}{v_F^2} + \frac{O(x^{12})}{8(E_F\tau)^2}, \tag{8.9}$$

[2] Note (see p. 54) that in the original basis diffuson has off-diagonal elements [(1,2) and (2,1)].

[3] The result (8.8) has been obtained in `TEKCT/SPIN-HALL/charge_cond.nb`. Later it has been confirmed by `CHARGE-COND/zla-{Drude,renorm}.max`.

so that conductivity tensor is isotropic, which contradicts to [49], where SOI is claimed to induce anisotropic contribution[4] to the charge conductivity, see ([49]49,50): $\delta\sigma \equiv |\sigma_{xy}|/\sigma_{xx} = 7ab/(4v_F^2)$. It also contradicts ([0508681]31) (where the correction is negative), and ([PRB67033104]19) (where the dimension is incorrect). Moral: be careful with the kinetic equations, see the end of §[48]VII.1. The dependence (8.8) coincides with (numerical) fig. [0510842]4.

Finally, I've also calculated the weak localization and found out that it does not produce any anisotropic contribution (at $\omega = 0$) within the prescribed accuracy.

8.3.2 SOI considered exactly

The calculation from Sec. 8.3.1 can be performed without assuming that $x_{a,b} \ll 1$ (i.e., without the diffusion approximation[5]). Let us use the results of sec. 6.5 for this. Using (7.13), we deduce that in the ZLA conductivity is given by

$$\frac{2\pi\hbar}{e^2}\tilde\sigma^{(0)}_{\alpha\beta} = \mathrm{Sp}\left[\hat v_\alpha \hat G_R \frac{\hat p_\beta}{m}\hat G_A\right] = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{p_\alpha p_\beta}{m^2}\left[g_r^- g_a^- + g_r^+ g_a^+\right] - \frac12\,\mathrm{Sp}\left[\left(\frac{e}{mc}\tilde A_\alpha\right)\frac{\hat p_\beta}{m}\hat M(g_r^- g_a^- - g_r^+ g_a^+)\right]. \tag{8.10}$$

Obviously, we have to face a usual divergence problem for the charge conductivity. As usually, [cf. the discussion in Sec. 8.1], we have to approximate $p \approx p_F$ in the current vertex in the main (Drude) contribution to the conductivity:

$$\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{p_\alpha p_\beta}{m^2}\left[g_r^- g_a^- + g_r^+ g_a^+\right] \approx \int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}n_\alpha n_\beta \int_0^\infty \frac{\mathrm{d}p}{2\pi}p\frac{p_F^2}{m^2}\left[g_r^- g_a^- + g_r^+ g_a^+\right] = \frac{\delta_{\alpha\beta}}{2}\left[2p_F l - \frac{2}{\pi} - \frac{1}{4p_F l}\right], \tag{8.11}$$

which is SOI independent. However, (8.11) is incorrect due to the following subtlety: I claim, that *only* in the divergent term[6] we are allowed to substitute $p \to p_F$. That is, $\int \frac{\mathrm{d}^2p}{(2\pi)^2}\frac{p_\alpha p_\beta}{m^2}\left[g_r^- g_a^- + g_r^+ g_a^+\right] = 2\frac{p_F^2}{m^2}\int \frac{\mathrm{d}^2p}{(2\pi)^2}n_\alpha n_\beta g_r^0 g_a^0$+SOI-dependent correction:

$$\begin{aligned}&\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}n_\alpha n_\beta\int_0^\infty\frac{\mathrm{d}p}{2\pi}\frac{p^3}{m^2}\left(g_r^- g_a^- + g_r^+ g_a^+ - 2g_r^0 g_a^0\right) = \frac{8l^2}{p_F l}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}n_\alpha n_\beta (p_S\tilde n)^2 = \\ &= 4m\tau\left[(a^2+b^2)\begin{pmatrix}1&0\\0&1\end{pmatrix} + ab\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right] = \frac{1}{2E_F\tau}\left[(x_a^2+x_b^2)\begin{pmatrix}1&0\\0&1\end{pmatrix} + x_a x_b\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right].\end{aligned} \tag{8.12}$$

Note that (8.12) has also sub-leading SOI-dependent terms, which I did not write since they are relatively small $\sim (p_F l)^{-1} \ll 1$ (the same is true for other equations in this section). The rest of (8.10) equals

$$\begin{aligned}&\frac12\,\mathrm{Sp}\left[\left(-\frac{e}{mc}\tilde A_\alpha\right)\frac{\hat p_\beta}{m}\hat M(g_r^- g_a^- - g_r^+ g_a^+)\right] = \frac{\alpha}{2}\sum_{i=1}^2\left(-\frac{e}{mc\alpha}\right)^2\mathrm{Sp}\left[\tilde A_\alpha n_\beta \tilde A_i\frac{n_i}{\tilde n}\cdot\frac{p}{m}(g_r^- g_a^- - g_r^+ g_a^+)\right] = \\ &= \delta_{\alpha\beta}\frac{\alpha}{2}\,\mathrm{Sp}\left[\left(-\frac{e}{mc\alpha}\tilde A_\alpha\right)^2\frac{n_\alpha^2}{\tilde n}\cdot\frac{p}{m}(g_r^- g_a^- - g_r^+ g_a^+)\right] = -2m\tau\left[(a^2+b^2)\begin{pmatrix}1&0\\0&1\end{pmatrix} + 2ab\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right],\end{aligned} \tag{8.13}$$

where we used (6.3) and the fact that

$$\int_0^\infty\frac{\mathrm{d}p}{2\pi}\frac{p^2}{m}\left(g_r^- g_a^- - g_r^+ g_a^+\right) = -\tilde n p_S\tau\left[4 - \frac{4}{p_F l} - \frac{1}{2(p_F l)^2}\right] \approx -4\tilde n m\alpha\tau.$$

We see that anisotropic terms in (8.12) and (8.13) cancel each other so that charge conductivity (8.10) is proportional to the unity tensor.

[4] Note that the ansatz for the kinetic equation used in [49], is not self-consistent, and this is accepted by the authors; so almost for sure the result ([49]49,50) is incorrect. in march 2006 DL told J. Schliemann about the result (8.9). In November 2006 J. Schliemann coauthored a paper where he gets a similar result ([0611328]27) for the conductivity: ([0611328](28)) is 4 times smaller than (8.9). I was unable to understand their derivation; I suspect that it is wrong. From p. 47 we see that in our case $\varphi = \pi/4$ so that in the original CS $\sigma = R_\varphi^{-1}\tilde\sigma R_\varphi$, $\varphi = \pi/4$.

[5] However, the assumption $x_{a,b} \ll 1$ is also necessary for the validity of the loop expansion. Означает ли это, что в случае сильного СОВ должна наступать Андерсоновская локализация?

[6] A calculation tip: first take integrals over ξ using residues, as if there were no problems with convergence at $|p| \to \infty$; then just take the real part of the integration result. This corresponds to substituting $p \to p_F$ in the divergent term and the correct treatment of the other terms. See also the discussion after (3.43).

8.3.3 Non-universal corrections

I've calculated the ZLA-bubble by $\vec{p}$ assuming finite $E_F\tau < \infty$, and this produced no corrections to our two-loop result. However, e.g., there are non-universal finite contributions to the diamagnetic current; how can we be sure that they are smaller than the two-loop contribution? The non-universal corrections to the WL can be even less controlled – there we could connect two different G_Rs in the HB with more than one DL. In short, for the moment I don't know how I could really be sure that all these non-universal corrections can not be of the same order, as our two-loop result. We only know that non-universal corrections can not cancel the universal ones.

In general (not only in ZLA), it would be nice if I could demonstrate that for all ME

$$\boxed{\text{precise GFB ME}} = \boxed{\text{approximate GFB ME}} \times \left(1 + \frac{a}{E_F\tau}\right), \quad \frac{\partial}{\partial x}\left(\frac{a}{E_F\tau}\right) = 0.$$

8.4 Two-loop diagrams

BTW, the complete list of diagrams is available in [83].

In previous sections (8.3.1) and 8.3.2, as well as in [49] we have seen, that the natural choice of precision is $\propto \sigma_D/(E_F\tau)^2$, see (8.9), which corresponds to that we have to consider diagrams till the two-loop approximation.[7] With my program (see sec. 12) I generate diagrams of the second loop and see that

- Here are the "skeleton" diagrams. After we expand all the HBs, we get in total 215 diagrams which can be divided in 7 groups. The most important diagrams are depicted in Fig. [20]**??**.
- A two-loop diagram with two CDs is generally smaller, as a two-loop diagram with three CDs, because the last one is "more divergent". To be more precise, they must be related as[8]

$$\int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{x^2+k^2} \frac{1}{x^2+q^2} \frac{1}{1+(k-q)^2l^2} \ll \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{x^2+k^2} \frac{1}{x^2+q^2} \frac{1}{x^2+(k-q)^2l^2}, \tag{8.14}$$

  where I've simplified the spin-structure of the diffuson, introducing mass x^2 in the most primitive way. By DL's request, I've demonstrated (by explicit calculation) in Sec. 8.7, we can neglect all diagrams except for [20]**??**, [20]**??**, and [20]**??**.
- It is a good move, before taking $\int d^2k/(2\pi)^2 \int d^2q/(2\pi)^2$, to introduce new integration variables: amplitude and anisotropy — as $P = K^2 + Q^2$ and $A = 2KQ/P$. This is possible since we know that the integrand is symmetric with respect to $K \leftrightarrow Q$, and $K, Q \geq 0$. Also the Jacobian is quite simple. The advantage is that an integrand, expressed in terms of P and A, automatically symmetric with respect to $K \leftrightarrow Q$. So, "false" divergences on small momenta vanish, see Sec. 8.5 and 8.6.

In order not to make calculations too complicated, I have to work in the DA, i.e., to assume that $x_{a,b} \ll 1$. Since the considered diagrams are already on the limit of our precision (2nd loop), no MSH-like problems (cf. Sec. 7.3) are

[7] One can on the other hand, conclude that the results (8.8) and (8.13) are incomplete since they do not satisfy the condition [66] that at $a = \pm b$ the conductivity has to be independent of the SOI.

[8] Note that this is without taking into account additional smallness $\sim q^*l \ll 1$ in the square GFBs of diagrams [20]**??**,[20]**??**,[20]**??**. This is the general rule VK told me: a diagram is inferior, if there is another diagram with the same number of loops, but with more CD lines. Use fig. 10.1 for memory refreshment. However, this argument is not obvious for the relation between [20]**??** and [20]**??**, since [20]**??** gains additional smallness because it has lonely vector vertices in its HBs, unlike [20]**??**.

α	0	0	1	1	2	2	3	3
β	1	3	1	3	1	3	1	3
γ	3	1	2	0	1	3	0	2
$\frac{L^x_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a+x_b)}$	-1	1	i	1	$-i$	$-i$	-1	i
$\frac{R^x_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a+x_b)}$	1	-1	$-i$	-1	i	i	1	$-i$

α	0	0	1	1	2	2	3	3
β	2	3	2	3	2	3	2	3
γ	3	2	2	3	1	0	0	1
$\frac{L^y_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a-x_b)}$	-1	1	i	i	$-i$	1	-1	$-i$
$\frac{R^y_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a-x_b)}$	1	-1	$-i$	$-i$	i	-1	1	i

Table 8.1: Non-zero elements of GFB [first order in (x_a, x_b)] of the diagram on fig. [20]?? for $q = k = 0$. (Saying that $q = k = 0$ in the HB, I get rid of its "classical", SOI-independent component; This "classical" component is non-zero, when $\sigma_\alpha\sigma_\beta\sigma_\gamma \propto \sigma_0$)

expected. (I mean, I can approximate $p \approx p_F$.) Analogously to sec. 7.4, the diagram on fig. [20]?? is equal to

$$\int \frac{d^2k}{(2\pi)^2}\int \frac{d^2q}{(2\pi)^2}\sum_{\alpha,\beta,\gamma=0}^{3}\sum_{\alpha',\beta',\gamma'=0}^{3} L_{\alpha\beta\gamma}D^{\alpha\alpha'}(\vec{k})D^{\beta'\beta}(\vec{k}+\vec{q})D^{\gamma\gamma'}(\vec{q})R_{\alpha'\beta'\gamma'} \tag{8.15}$$

$$L = L^{(0)} + L^{(1)} + L^{(2)}, \quad L^{(0)}_{\alpha\beta\gamma} = \operatorname*{Sp}_p\left[G^<(\vec{p})\sigma_\alpha G_A(\vec{p}-\vec{k})\sigma_\beta G_R(\vec{p}+\vec{q})\sigma_\gamma\right], \tag{8.16}$$

$$L^{(1)}_{\alpha\beta\gamma} = \frac{1}{4\pi\nu\tau}\sum_{\delta=0}^{3}\operatorname*{Sp}_p\left[G^<(\vec{p})\sigma_\alpha G_A(\vec{p}-\vec{k})\sigma_\delta\right]\operatorname*{Sp}_p\left[G_A(\vec{p})\sigma_\delta G_A(\vec{p}-\vec{k})\sigma_\beta G_R(\vec{p}+\vec{q})\sigma_\gamma\right],$$

$$L^{(2)}_{\alpha\beta\gamma} = \frac{1}{4\pi\nu\tau}\sum_{\delta=0}^{3}\operatorname*{Sp}_p\left[G^<(\vec{p})\sigma_\delta G_R(\vec{p}+\vec{q})\sigma_\gamma\right]\operatorname*{Sp}_p\left[G_R(\vec{p})\sigma_\alpha G_A(\vec{p}-\vec{k})\sigma_\beta G_R(\vec{p}+\vec{q})\sigma_\delta\right],$$

$$R = R^{(0)} + R^{(1)} + R^{(2)}, \quad R^{(0)}_{\alpha\beta\gamma} = \operatorname*{Sp}_p\left[G^>(\vec{p})\sigma_\alpha G_R(\vec{p}+\vec{k})\sigma_\beta G_A(\vec{p}-\vec{q})\sigma_\gamma\right],$$

$$R^{(1)}_{\alpha\beta\gamma} = \frac{1}{4\pi\nu\tau}\sum_{\delta=0}^{3}\operatorname*{Sp}_p\left[G^>(\vec{p})\sigma_\delta G_A(\vec{p}-\vec{q})\sigma_\gamma\right]\operatorname*{Sp}_p\left[G_A(\vec{p})\sigma_\alpha G_R(\vec{p}+\vec{k})\sigma_\beta G_A(\vec{p}-\vec{q})\sigma_\delta\right],$$

$$R^{(2)}_{\alpha\beta\gamma} = \frac{1}{4\pi\nu\tau}\sum_{\delta=0}^{3}\operatorname*{Sp}_p\left[G^>(\vec{p})\sigma_\alpha G_R(\vec{p}+\vec{k})\sigma_\delta\right]\operatorname*{Sp}_p\left[G_R(\vec{p})\sigma_\delta G_R(\vec{p}+\vec{k})\sigma_\beta G_A(\vec{p}-\vec{q})\sigma_\gamma\right].$$

At $q = k = 0$ we notice that, e.g., $L^{(1)} = \pm R^{(2)}$ [since the corresponding expressions differ only by the sign of τ; however, from the simplest example of $\int d^dp/(2\pi)^d g_r(\vec{p})g_a(\vec{p}) = 2\pi\nu|\tau|$ we know that sometimes τ enters under the "module" sign into the result of the integration.]

It is quite easy to demonstrate that $\sigma_{xy} = \sigma_{yx} = 0$. Let us now see if $\sigma_{xx} = \sigma_{yy}$. Let us introduce a projection and index transposition operation C_{xy} for indices α, β, γ in (8.15), defined according to the rule:

$$\begin{gathered}\text{The definition of } C_{xy}: \text{ it exchanges } Q_x \leftrightarrow Q_y,\ K_x \leftrightarrow K_y,\\ \text{and, among indices } \alpha,\beta,\gamma,\alpha',\beta',\gamma', \quad C_{xy} \text{ exchanges } 1 \leftrightarrow 2, \text{ e.g.,}\\ C_{xy}[R_{123}(q_x,q_y,k_x,k_y)] = R_{213}(q_y,q_x,k_y,k_x),\\ C_{xy}[R_{223}(q_x,q_y,k_x,k_y)] = R_{113}(q_y,q_x,k_y,k_x), \quad C_{xy}[R_{323}(q_x,q_y,k_x,k_y)] = R_{313}(q_y,q_x,k_y,k_x).\end{gathered} \tag{8.17}$$

The following property has been checked up to the precision of $(x_a, x_b, lq, lk)^6$:

$$L^x_{\alpha\beta\gamma}(\vec{q},\vec{k},x_a,x_b) = C_{xy}\left[L^y_{\alpha\beta\gamma}(-\vec{q},-\vec{k},x_a,-x_b)\right]^*, \quad R^x_{\alpha\beta\gamma}(\vec{q},\vec{k},x_a,x_b) = C_{xy}\left[R^y_{\alpha\beta\gamma}(-\vec{q},-\vec{k},x_a,-x_b)\right]^*. \tag{8.18}$$

If we forget about elements of HBs, which are multiplied by zeros from the diffuson matrices, then we get better symmetries for the product of the HBs (checked up to the 6th order):

$$\begin{gathered}L^x_{\alpha\mu\gamma}(x_a,x_b)R^x_{\alpha'\mu'\gamma'}(x_a,x_b) = C_{xy}[L^y_{\alpha\mu\gamma}(x_a,-x_b)R^y_{\alpha'\mu'\gamma'}(x_a,-x_b)],\\ L^x_{\alpha\mu\gamma}(\vec{k},\vec{q},x_a,x_b)R^x_{\alpha'\mu'\gamma'}(\vec{k},\vec{q},x_a,x_b) = C_{xy}[L^y_{\alpha\mu\gamma}(-\vec{k},-\vec{q},-x_a,x_b)R^y_{\alpha'\mu'\gamma'}(-\vec{k},-\vec{q},-x_a,x_b)].\end{gathered} \tag{8.19}$$

From (6.28) and (8.19) we conclude that, as we predicted in Sec. 8.2,

$$\sigma_{xx}(x_a = 0, x_b) = \sigma_{yy}(x_a = 0, x_b), \quad \sigma_{xx}(x_a, x_b = 0) = \sigma_{yy}(x_a, x_b = 0). \tag{8.20}$$

α	0	0	1	1	2	2	3	3
β	1	3	1	3	1	3	1	3
γ	3	1	2	0	1	3	0	2
$\frac{L^x_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a+x_b)}$	-1	1	i	1	$-i$	$-i$	-1	i
$\frac{R^x_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a+x_b)}$	0	0	$-2i$	0	$2i$	$2i$	0	$-2i$

α	0	0	1	1	2	2	3	3
β	2	3	2	3	2	3	2	3
γ	3	2	2	3	1	0	0	1
$\frac{L^y_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a-x_b)}$	-1	1	i	i	$-i$	1	-1	$-i$
$\frac{R^y_{\alpha\beta\gamma}}{4\pi\nu v_F\tau^3(x_a-x_b)}$	0	0	$-2i$	$-2i$	$2i$	0	0	$2i$

Table 8.2: The same, as in Tab. 8.1, but renormalized with diagrams [20]?? and [20]??. We see that now MEs of R, which could lead to divergences, vanish.

8.5 About divergences

We know that $D^{00}(k) \propto 2/(lk)^2$ is not affected by SOI, and, in principle, can lead to divergences in $\int d^2k/(2\pi)^2 \int d^2q/(2\pi)^2$ at small momenta in our expression for the diagram [20]??. Let us at first consider MEs of HBs from Tab. 8.1. For $lk \ll \min(|x_a|, |x_b|)$, integrand contains four terms $\propto D^{00}(k)$:

$$D^{00}(k)\left[D^{13}(\vec{k}+\vec{q})D^{31}(\vec{q}) + D^{31}(\vec{k}+\vec{q})D^{13}(\vec{q}) - D^{11}(\vec{k}+\vec{q})D^{33}(\vec{q}) - D^{33}(\vec{k}+\vec{q})D^{11}(\vec{q})\right] \approx$$
$$\approx 2D^{00}(k)\left[D^{13}(\vec{q})D^{31}(\vec{q}) - D^{11}(\vec{q})D^{33}(\vec{q})\right]$$

First let us explore the case, when both $lk, lq \ll \min(|x_a|, |x_b|)$. In this limit we use expression (6.37) for the diffuson. The most dangerous MEs of the HBs are those at $k = q = 0$. Their approximate values are summarized in Tab. 8.1. With these *approximate* values we get divergences

$$\sigma_{xx}(x_a, x_b) - \sigma_{yy}(x_a, x_b) \propto x_b^4 \int \frac{d^2q}{(2\pi)^2} D^{00}(q)\left[D^{11}(x_a+x_b)^2 - D^{22}(x_a-x_b)^2\right] + \text{other terms.}$$

There would be no divergence, if *exact* values of $L^{x,y}$ and $R^{x,y}$ at $k = q = 0$ would be such that, e.g.,

$$\text{at } k = q = 0 \quad D^{11}L^x_{013}R^x_{013} - D^{22}L^y_{023}R^y_{023} = 0, \text{ or } \quad (1 - X_D^{22})L^x_{013}R^x_{013} - (1 - X_D^{11})L^y_{023}R^y_{023} = 0, \tag{8.21}$$

which I've checked up to the 6th order, see `2nd_loop/getSymmetries.max` (I am sure it is valid also for other necessary indices.) So apparently we have no divergence when both k, q are small.

What happens if only one of $\vec{k}, \vec{q}$ is not small? Then we have divergences in diagram [20]??. Let us consider the case of zero MF. Then in addition to our "main" diagram [20]?? (which does not contain Cooperons) we have to consider the contribution of other diagrams. Let us focus on the group of diagrams [20]??, [20]??, [20]??. Summing them all is equivalent to a sort of "renormalization" of the right Hikami box of [20]?? (compare Tab. 8.1 with Tab. 8.2): $R^x_{013} = R^x_{031} = R^x_{130} = R^x_{310} = R^y_{023} = R^y_{032} = R^y_{230} = R^y_{320} = 0$, while all other MEs of R are doubled (checked up to the 2nd order). This divergence cancellation occurs according to the theorem on p. [84]4677.

Note, however, that in the automatic calculation of diagrams spin indices and momentum variables in diagrams [20]??, [20]??, and [20]?? do not coincide, so that there are non-zero MEs corresponding to $D^{00}_{\vec{k}}$ and $D^{00}_{\vec{q}}$. This apparently results in divergences at $\vec{k}, \vec{q} \to 0$. These divergences are "false"; they get mutually cancelled. A good way to get rid of them is the usage of symmetric momenta variables (P, A) defined in Sec. 8.6.

Next, looking on expansion, we understand that we risk to get divergences at $lk, lq \gg 1$, because we expand our HBs assuming $lk, lq \ll 1$ (see Sec. 13.2). However, the appearance of such a divergence would mean that (l, q) are not small momentum variables, which would contradict at list the assumption that we work in the diffuson approximation (and probably even the assumption of the loop expansion in Sec. 3.6.) Fortunately, in my calculation it did not occur.

Finally, for $m \geq 2$ in (8.6), one may face divergences at $k, q \to 0$, if he forgets subtracting the WL-alike diagram with 3 dashed lines, see the caption for Fig. [20]??. One may naïvely think that this "extra" diagram [which we must 3 times subtract from the sum of [20]??,[20]??,[20]??] can be obtained from any of [20]??,[20]??,[20]?? by substituting $\bar{X}_D \to 0$ into the expression (??) for their diffusons. Viewed in this way, this subtraction seems unimportant since it can not affect terms with $m < 6$ from (8.6). However, one should also take into account the change in momentum structure of the diagram. This leads to the fact that it has only one loop [instead of two loops for [20]??,[20]??,[20]??], so it can affect only terms with $m \geq 2$. Note that the *infinite* WL-series, in principle, could affect *all* terms in the expansion (8.6) [but not S_{00} as we've concluded in Sec. 8.7.]

8.6 Integrating $\int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2}$ on computer: analytics and numerics

See `diagrams.max/EXAMPLES/conductivity-SOI/5/` We calculate the leading term $\propto S_{00}$ in the expansion (8.6). In the calculation, we use dimensionless HBs and dimensionless diffusons, and we integrate in dimensionless variables, so we should not forget that

$$\text{The final result} = \frac{e^2}{h}(2m\tau^3 v_F x)^2 \delta \frac{x^4}{l^4} \frac{1}{(2m\tau x^2)^3} \frac{1}{(2\pi)^2} \int_0^\infty dK \int_0^\infty dQ \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{2\pi} \frac{d\psi}{2\pi} \times \\ \times \text{our dimensionless expression,} \tag{8.22}$$

where the обезразмеривающий factor $(2m\tau^3 v_F x)^2$ is introduced manually in `integrand.max`. Further change of variables brings us additional coefficients, see (8.23).

Since the angular dependence of the denominator is only due to $(\vec{K}+\vec{Q})^2 = K^2 + Q^2 + 2KQ\cos\psi$, we can easily perform the integration on the other angle. Then we have to integrate the rational function

$$\int_0^{2\pi} \frac{d\psi}{2\pi} \frac{P_1(\sin\psi, \cos\psi)}{P_2(\cos\psi)},$$

where $P_{1,2}$ are polynomials. The denominator is even in ψ, so we can leave only even (in ψ) part of the numerator. Then numerator thus can be expressed as a another polynomial:

$$\int_0^{2\pi} \frac{d\psi}{2\pi} \frac{P_3(\cos\psi)}{P_2(\cos\psi)}, \quad P_3(\cos\psi) = \sum_n a_n \cos^n \psi.$$

We perform the integration over ψ using the method from p. [85]228-229. Because of the large number of terms to integrate and large size of the expressions, this analytical integration (i.e., basically, calculation of residues) is done on computer, see `integrate.max`. We use the fact that the denominators of all massfull elements of the diffuson $D_{\vec{k}+\vec{q}}$ can be factorized into two expressions: $uno = 1 + (\vec{k}+\vec{q})^2$ and $due = 2 - (\vec{k}+\vec{q})^2 + (\vec{k}+\vec{q})^4$. The size of the integration result grows rapidly with powers of *uno* and *due* in the denominator, so it is neccessary to split integrands into elementary fractions.[9]

Since (according to the basic assumption of the loop expansion) integrals over $(\vec{k},\vec{q})$ converge on the scale of $k \lesssim x/l$ and $q \lesssim x/l$, we naturally introduce dimensionless variables $K = kl/x$ and $Q = ql/x$. Next, the integrand is symmetric with respect to $K \leftrightarrow Q$, and the integration operator has this symmetry too. So we symmetrize[10] every term of the integrand, and express it in terms of new variables[11] $P = K^2 + Q^2$ and $A = 2KQ/P$. Accordingly, our integration operator is changed:

$$\int_0^\infty dK \int_0^\infty dQ\, KQ = 2\int_0^\infty dK \int_0^K dQ\, KQ = 2\int_0^\infty dP \int_0^1 dA \frac{PA}{2J}, \\ A = \frac{2KQ}{K^2+Q^2}, \quad J = \left|\frac{\partial(P,A)}{\partial(K,Q)}\right| = \left|4\frac{K^2-Q^2}{K^2+Q^2}\right| = 4\sqrt{1-A^2}. \tag{8.23}$$

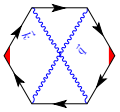
[9] Huge integration results usually contain the difference of large numbers, which is calculated numerically with bad accuracy, which results in various artefacts, like, e.g., "false" divergences.

[10] With this symmetrization we get rid of "false" divergences at $k, q \to 0$, see Sec. 8.5.

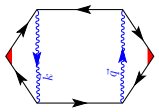
[11] Since the vicinity of $A = 1$ is very important for the integration, it makes sense to further change variable $A = 1 - B^{-2}$, see here for more details.

8.7 Other diagrams

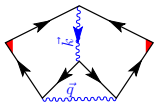
By the request of DL, I've checked that [20]**??**-alike looking hexagonals (see `hexagons/`), and saw that their contribution[12] to the expansion (8.6) is (exactly) $S_{00} = 0$:



$$= \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \sum_{\alpha,\beta,\gamma,\delta=0}^{3} D_{\vec{k}}^{\alpha\gamma} D_{\vec{q}}^{\delta\beta} \operatorname*{Sp}_{\vec{p}} \left[G^<(\vec{p}) \tilde{\sigma}_\alpha G_A(\vec{p}-\vec{k}) \tilde{\sigma}_\beta G^>(\vec{p}-\vec{k}+\vec{q}) \tilde{\sigma}_\gamma G_R(\vec{p}+\vec{q}) \tilde{\sigma}_\delta \right],$$



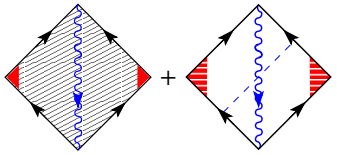
$$= \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \sum_{\alpha,\beta,\gamma,\delta=0}^{3} D_{\vec{k}}^{\alpha\delta} D_{\vec{q}}^{\gamma\beta} \operatorname*{Sp}_{\vec{p}} \left[G^<(\vec{p}) \tilde{\bar{\sigma}}_\alpha^\dagger G_A^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_\beta G^>(\vec{p}-\vec{k}+\vec{q}) \tilde{\bar{\sigma}}_\gamma^* G_R^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_\delta^T \right],$$



$$= \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \sum_{\alpha,\beta,\gamma,\delta=0}^{3} D_{\vec{k}}^{\alpha\beta} D_{\vec{q}}^{\delta\gamma} \operatorname*{Sp}_{\vec{p}} \left[G^<(\vec{p}) \tilde{\bar{\sigma}}_\alpha^\dagger G^{>T}(\vec{k}-\vec{p}) \tilde{\sigma}_\gamma^T G_A^T(\vec{k}-\vec{q}-\vec{p}) \tilde{\bar{\sigma}}_\beta G_R(\vec{p}+\vec{q}) \tilde{\sigma}_\delta \right].$$

In principle, one can add lonely IALs into their GFBs, but this will separate current vertices, and will make the argument (8.14) valid. I mean, we can forget about these lonely IALs. STOP No, this is not true!!! (since, e.g, I can insert IALs into diagram [20]**??** without splitting vertices)

In addition, there is a WL-diagram[13] together with its two-loop correction:

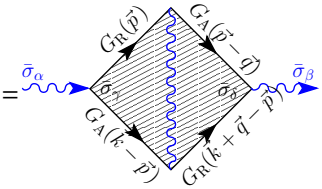


$$= \int \frac{d^2q}{(2\pi)^2} \sum_{\alpha,\beta=0}^{3} \left(D_{\vec{q}}^{\alpha\beta} + \mathbb{C}_{\vec{q}}^{\alpha\beta} \right) \left\{ \operatorname*{Sp}_{\vec{p}} \left[G^<(\vec{p}) \tilde{\bar{\sigma}}_\alpha^\dagger G^{>T}(\vec{q}-\vec{p}) \tilde{\bar{\sigma}}_\beta^T \right] + \right.$$

$$+ \frac{1}{4\pi\nu\tau} \sum_{\gamma=0}^{3} \operatorname*{Sp}_{\vec{p}} \left[G^<(\vec{p}) \tilde{\sigma}_\gamma G_R^T(\vec{q}-\vec{p}) \tilde{\bar{\sigma}}_\beta^T \right] \operatorname*{Sp}_{\vec{p}} \left[G_R(\vec{p}) \tilde{\bar{\sigma}}_\alpha^\dagger G^{>T}(\vec{q}-\vec{p}) \tilde{\sigma}_\gamma \right] +$$

$$\left. + \frac{1}{4\pi\nu\tau} \sum_{\gamma=0}^{3} \operatorname*{Sp}_{\vec{p}} \left[G^<(\vec{p}) \tilde{\bar{\sigma}}_\alpha^\dagger G_A^T(\vec{q}-\vec{p}) \tilde{\sigma}_\gamma \right] \operatorname*{Sp}_{\vec{p}} \left[G_A(\vec{p}) \tilde{\sigma}_\gamma G^{>T}(\vec{q}-\vec{p}) \tilde{\bar{\sigma}}_\beta^T \right] \right\},$$

where $\mathbb{C}_{\vec{q}}^{\alpha\beta}$ is the $(p_F l)^{-1}$-correction to the cooperon:

$\mathbb{C}_{\vec{q}}^{\alpha\beta} =$ 

$$= \sum_{\gamma,\delta,\mu,\nu=0}^{3} D_{\vec{k}}^{\alpha\gamma} D_{\vec{k}}^{\delta\gamma} \int \frac{d^2q}{(2\pi)^2} D_{\vec{q}}^{\mu\nu} \times$$

$$\times \left\{ \operatorname*{Sp}_{\vec{p}} \left[G_A^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_\gamma^\dagger G_R(\vec{p}) \tilde{\sigma}_\mu G_A(\vec{p}-\vec{q}) \tilde{\bar{\sigma}}_\delta^T G_R^T(\vec{k}+\vec{q}-\vec{p}) \tilde{\sigma}_\nu^T \right] + \right.$$

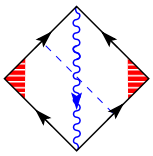
$$+ \frac{1}{4\pi\nu\tau} \sum_{\kappa=0}^{3} \operatorname*{Sp}_{\vec{p}} \left[G_A^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_\gamma^\dagger G_R(\vec{p}) \tilde{\sigma}_\kappa G_R^T(\vec{k}+\vec{q}-\vec{p}) \tilde{\sigma}_\nu^T \right] \operatorname*{Sp}_{\vec{p}} \left[\tilde{\sigma}_\kappa G_R(\vec{p}) \tilde{\sigma}_\mu G_A(\vec{p}-\vec{q}) \tilde{\bar{\sigma}}_\delta^T G_R^T(\vec{k}+\vec{q}-\vec{p}) \right] +$$

$$\left. + \frac{1}{4\pi\nu\tau} \sum_{\kappa=0}^{3} \operatorname*{Sp}_{\vec{p}} \left[G_A^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_\gamma^\dagger G_R(\vec{p}) \tilde{\sigma}_\mu G_A(\vec{p}-\vec{q}) \tilde{\sigma}_\kappa \right] \operatorname*{Sp}_{\vec{p}} \left[\tilde{\sigma}_\kappa G_A(\vec{p}-\vec{q}) \tilde{\bar{\sigma}}_\delta^T G_R^T(\vec{k}+\vec{q}-\vec{p}) \tilde{\sigma}_\nu^T G_A^T(\vec{k}-\vec{p}) \right] \right\}.$$

[12] Note that, apparently, the corresponding $S_{20} \neq 0$ – at least before the integration over the modulus of momenta k, q.

[13] Note that the famous "WL logarithm" cancels away from the *difference* $\sigma_{xx} - \sigma_{yy}$ I had to be anxious about WL diagrams, since I was not sure that they don't contain convergent (at small momenta) terms.

Finally, there is one more diagram of the WL-type:



$$= \sum_{\alpha=0}^{3} L_\alpha D_0^{\alpha\alpha} R_\alpha, \quad L_\alpha = \int \frac{d^2k}{(2\pi)^2} \sum_{\beta,\beta'=0}^{3} D_{\vec{k}}^{\beta\beta'} \Big\{ \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G^<(\vec{p}) \tilde{\bar{\sigma}}_\beta^\dagger G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\alpha^T G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_{\beta'}^T \Big] +$$

$$+ \frac{1}{4\pi\nu\tau} \sum_{\delta=0}^{3} \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G^<(\vec{p}) \tilde{\sigma}_\delta G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_{\beta'}^T \Big] \mathop{\mathrm{Sp}}_{\vec{p}} \Big[\tilde{\sigma}_\delta G_{\mathrm{R}}(\vec{p}) \tilde{\bar{\sigma}}_\beta^\dagger G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\alpha^T G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \Big] +$$

$$+ \frac{1}{4\pi\nu\tau} \sum_{\delta=0}^{3} \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G^<(\vec{p}) \tilde{\bar{\sigma}}_\beta^\dagger G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\delta \Big] \mathop{\mathrm{Sp}}_{\vec{p}} \Big[\tilde{\sigma}_\delta G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\alpha^T G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_{\beta'}^T G_{\mathrm{A}}(\vec{p}) \Big] \Big\},$$

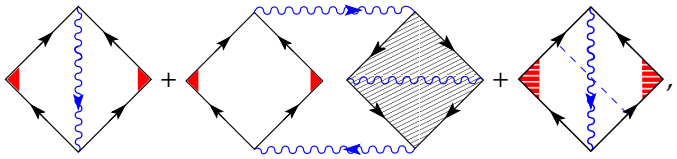
$$R_\alpha = \int \frac{d^2k}{(2\pi)^2} \sum_{\beta,\beta'=0}^{3} D_{\vec{k}}^{\beta\beta'} \Big\{ \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G^> \tilde{\bar{\sigma}}_\beta^* G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\alpha^T G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_{\beta'} \Big] +$$

$$+ \frac{1}{4\pi\nu\tau} \sum_{\delta=0}^{3} \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G^> \tilde{\sigma}_\delta G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_{\beta'} \Big] \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G_{\mathrm{A}}(\vec{p}) \tilde{\bar{\sigma}}_\beta^* G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\alpha^T G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\delta \Big] +$$

$$+ \frac{1}{4\pi\nu\tau} \sum_{\delta=0}^{3} \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G^> \tilde{\bar{\sigma}}_\beta^* G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\delta \Big] \mathop{\mathrm{Sp}}_{\vec{p}} \Big[G_{\mathrm{R}}(\vec{p}) \tilde{\sigma}_\delta G_{\mathrm{R}}^T(\vec{k}-\vec{p}) \tilde{\sigma}_\alpha^T G_{\mathrm{A}}^T(\vec{k}-\vec{p}) \tilde{\bar{\sigma}}_{\beta'} \Big] \Big\}.$$

The calculation in `WL/2.max` shows that $L_\alpha \neq R_\alpha$.

From the above diagrams, we are afraid of the following:[14]



where we have to calculate the HBs explicitly and demonstrate that they can not alter S_{00} in the expansion (8.6). In the first diagram, the zeroth-order (in SOI) of the HB is important, which is independent on the cooperon indices so that the diagram is proportional to $C^{00} - C^{11} - C^{22} - C^{33}$, which is isotropic (STOP I wonder if this isotropy is exact). The second diagram is almost the same as the first one, but with the renormalized cooperon. As for the 3rd diagram, by using brutal computer force I've demonstrated in `WL/2.max` that it is also harmless. I used the fact that only diagonal MEs of the diffuson survive the angular integration over its momentum.

Later I've made an independent direct check[15]: divided all 215 diagrams (yes, 215 ones, if we consider every HB as a sum of three!) into 7 groups, and demonstrated, that only one (fifth) group of 27 diagrams [shortly represented by three diagrams in Fig. [20]**??**, [20]**??**, and [20]**??**] gives rise to S_{00} in (**??**).

8.8 Finite-frequency corrections

B. Altshuler: the result for the conductivity at $\omega = 0$ in (**??**) is singular at $a^2 + b^2 = 0$, so it would be interesting to calculate it in case of (large in comparison with SOI) finite frequency. The calculation for this case ($x^2 \ll -i\omega\tau$) is performed in `diagrams.max/EXAMPLES/conductivity-SOI/wt_greater_x/` The integrand is much simpler, than for $\omega = 0$ – See [66, 20]. The case $x^2 \gg -i\omega\tau$ is much more sophisticated; for the moment it is not ready yet.

We studied this finite-frequency case because this is the simplest way to take the dephasing into account: substituting $-i\omega\tau \to \tau/\tau_\varphi$ – see, e.g., before ([7]6.24). Note, however, that Montambaux [1] claims that the diffuson is a classical object.[16]

See Sec. 11.5.2 for the estimate of τ_φ.

[14] If I had to calculate higher in x orders of the diagrams in this section, I would have to subtract some diagrams with a finite number of IALs, since I start my cooperons with *one* IAL.

[15] See `diagrams.max/EXAMPLES/conductivity-SOI/`

[16] STOP Discuss this with DL: In particular he says that classical conductance is proportional to the diffuson. I understand this as a claim that diffuson does not die in a large sample at room temperature.

8.9 Conclusions

The coefficient S_{00} gives the anisotropic part of the conductivity tensor of a *phase coherent* semiconductor, and is valid[17] for $x > l/L_\varphi(T)$, where $L_\varphi(T)$ is the phase coherence length. In case when $L_\varphi = \infty$ the correction (15) depends non-analytically on the SOI amplitudes in the vicinity of $x = 0$. At $x = 0$ (i.e., in the absence of SOI) the conductivity tensor is isotropic but [at $T = 0$, if we believe that[18] $L_\varphi(T = 0) = \infty$] an infinitesimal SOI brings it to the anisotropic phase. We expect that this spontaneous symmetry breaking does not take place for $T > 0$, because of the finite value of $L_\varphi(T)$. Thus at $T = 0$ we observe a sort of a phase transition,[19] which can be described by the following 2D order parameter:

$$\vec{\eta} = (\mathrm{Sp}\,[\sigma_1\sigma]\,, \mathrm{Sp}\,[\sigma_3\sigma])\,.$$

Differently to a superconductor or a ferromagnet, the order parameter $\eta(T)$ has infinite support. So a closer analogy for us is the Bose-Einstein condensation (=BEC), where the order parameter is $\frac{\text{number of particles in the ground state}}{\text{total number of particles}}$. However, in BEC there is a non-analicity in the $\mu(T)$-dependence at the transition temperature[20]. I don't know if we have such non-analyticities in our system.

The rotation in the coordinate space corresponds to the rotation in the $\vec{\eta}$-space by double angle. While the magnitude η is zero for the system without SOI, an arbitrary small SOI makes it finite provided that $L_\varphi = \infty$ and the "harmonic average" δ of Rashba and Dresselhaus amplitudes is non-zero. The direction of $\vec{\eta}$ is then determined by the direction of crystal axes, determining the "main axes" of the SOI part of the Hamiltonian. The isotropic phase in the absence of SOI can be restored by destructing the phase coherence, e.g., by increasing the temperature.

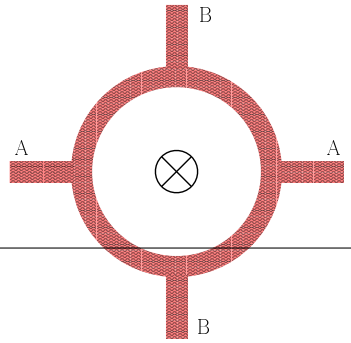


Figure 8.1: This figure is obsolete – see my poster. The measuring device. The voltage is applied between two A-contacts. The anisotropic conductivity results in the non-zero potential difference between two B-contacts, depending on the piercing magnetic flux.

In conclusion, it is not clear:

- Do we have a phase transition or a "crossover"?
- Is the transition temperature $T = 0$ or $T^* > 0$? [T^* being the scale of the $L_\varphi(T)$ dependence.]
- Do we have non-analyticities (at the transition point) in the dependencies of any thermodynamic parameters of the system?

For the information about BEC, see §[38]45, p. [45]16 and pp. 219-220 from the first volume of Russian physical encyclopedia.

DL: "the classification of different strange phase transitions can be found in the (in my opinion – boring) book by Subir Sachdev." I understood that this classification is more like a play of words; this classification does not have deep physical sense.

[17] Following the request of B. Altshuler (supported by DL), I've calculated the divergence of $\delta\sigma(\omega \neq 0)$ here.

[18] In Fig. 0707.1993 1 it is claimed that L_φ becomes infinite at finite (though very small) temperature.

[19] In fact, according to[86], the main feature of a phase transition consists in the following: introduction of an infinitesimal term in the Hamiltonian results in a finite correction to physical characteristics of the system.

[20] which results in non-analyticity of other thermodynamic parameters of the system. Note that this non-analyticity is usually drawn approximately, so that $\frac{\partial\mu(T)}{\partial T}$ is a discontinous function. I suspect that in reality all the derivatives of the order parameter and $\mu(T)$ are everywhere continuous. Можно попробовать это доказать с помощью теоремы о неявной функции.

Part III

Current autocorrelator

On p. 9 of cond-mat/0201007, they say, that "The phase coherence is not significant for the current fluctuations in the weakly disordered metals". In [87] they say that inelastic phonon scattering processes kills shot noise, so that in the macroscopic piese of metal it is not observed. On the other hand, inelastic elecron-electron interaction even slightly enhances it!

When the electron-phonon interaction is weak and the thermoconductivity is sufficient, everywhere in the sample the energy distribution f_E is just a shifted equilibrium one, with a width corresponding to the surrounding temperature. In this situation, the consideration based on the Landauer formalism must fail (see sec. **??**), and one has to use quantum kinetic equation instead.

Now suppose there is no electron-phonon interaction, but there is strong elecron-electron interaction. Then f_E is again a Fermi function, but its width is given by an affective temperature (9.38). This should also give no contribution to the shot noise, but should enhance Nycquist noise (according to sec. **??**).

Sec. **??** demonstrates that splitting noise in Nyquist (equilibrium) noise and shot noise is completely artificial. In a homogeneous quasi-1D (two-terminal) sample the noise is determined by the effective temperature; the latter depends on whether the system is in equilibrium or not.

8.10 A note about different types of averaging

One can be confused by the fact that we have three types of averaging that we have: quantum, statistical, and disorder averaging. In the definition of the the physical quantity, like concentration or current, the first two averages already present. We can thus calculate only a disorder-correlator, and this would be just the same, as in UCF or with my current-current correlator from thesis [88]. However, we could also calculate a correlator with all the three averages involved. Such a correlator will be expressed via the 2-particle density matrix.

Thus one has to distinguish two different types of correlators. The first ones are "normal", which are in the textbooks, where no disorder averaging technique is discussed. A "normal" current-current correlator, e.g. gives us information about the current fluctuatuins in one particular sample.

We are always interested also in averaging over the ensemble of samples. We calculate UCF via neglecting thermodynamical fluctuations. And we are right to do so since conductivity has no thermodynamical fluctuations by definition. In case of current-current correlator this depends on what we want to calculate. We may want to separate mesoscopic fluctuations from thermodynamic; then the calculation goes like in [88] or like in case of UCF.

What type of correlator do we need when studying noise? –The complete one, *including* thermodynamic averaging, since we are interested in the spin-correlations in one particular disordered sample.

8.11 Literature

См. Письма в ЖЭТФ т.49 стр. 513.

8.12 Only disorder averaging


См. также мою дисертацию [88]. Счёт третьего момента тока, притом с учётом взаимодействия сделан в cond-mat/0503552.

When we don't take interaction into account, a physical system can not distinguish between the equilibrium and non-equilibrium. In real physical systems interaction always present, but in case of equilibrium systems it sometimes can be neglected - an approximation which is much more questionable in case of a non-equilibrium system. In [38] on page 82 it is stated that "before relaxation processes finish" the interaction cannot be neglected, no matter how small it is. But this was told about a non-equilibrium system relaxing into equilibrium state. Is it true for a system in a *non-equilibrium steady state*? I think - yes, but I am not sure. As an example, let us study current-current correlator without interaction between the electrons.

8.13 Current in case of no interaction between electrons

Let us derive at first more convenient formula for the current in the coordinate space. We consider equilibrium system without interaction between the electrons, so that we have one-particle energy spectrum; Due to this we can write

$$\vec{j} = \sum_n f_T(E_n)\vec{j}_n, \tag{8.24}$$

where $f_T(E_n)$ is the average number of particles in a state with given energy E_n. Now let us obtain (8.24) formally. An arbitrary state $\psi = \sum_n C_n\varphi_n$ produces the current

$$j(r,t) \propto \sum_{mn} C_n C_m^* (\varphi_n \nabla\varphi_m^* - \varphi_m^* \nabla\varphi_n). \tag{8.25}$$

We want now to average (8.25) over the ensemble of the systems, that is, to use the density matrix $\rho^1 \equiv \langle C_n C_m^* \rangle$. For simplification we now make an unnecessary assumption that the spectrum is non-degenerate. Then the off-diagonal elements of $\langle C_n C_m^* \rangle$ are zeros, which proves (8.24).

From (3.2), ([19]2.7), ([19]2.8), ([19]3.16), ([19]3.11), (13.19) and (13.25) we see that

$$\vec{j} = \lim_{\vec{r}'\to\vec{r}} \left[\left(-\frac{ie\hbar}{2m}\left(\vec{\nabla}_{\vec{r}} - \vec{\nabla}_{\vec{r}'}\right) - \frac{e^2\hbar^2}{2m}\vec{A} \right) K'(\vec{r},\vec{r}';E) \right], \tag{8.26}$$

$$K'(\vec{r},\vec{r}';E) = e^{-E/T} J(\vec{r},\vec{r}';E), \quad J(\vec{r},\vec{r}';E) = \frac{1}{2\pi}\frac{i}{1+e^{-E/T}} \left(G_{\mathrm{R}}(\vec{r},\vec{r}';E) - G_{\mathrm{A}}(\vec{r},\vec{r}';E) \right).$$


Внимание: неправильная размерность в (8.26)!

There should be no contradiction whether to define the current from K or K' due to ([19]2.9). Note that here I did not use the fact that we have no interaction.

Finally we have

$$\vec{j}(\vec{r}) = \frac{e\hbar}{2m}\int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} f_T(E) \lim_{\vec{r}'\to\vec{r}} (\vec{\nabla}_{\vec{r}} - \vec{\nabla}_{\vec{r}'} - 2ie\vec{A}) \left[G_{\mathrm{R}}(\vec{r},\vec{r}') - G_{\mathrm{A}}(\vec{r},\vec{r}') \right], \quad f_T(E) = \frac{1}{1+e^{E/T}}. \tag{8.27}$$

From (3.30) it follows that in case of no interaction in (8.27) the integrand = 0 for $E < -\mu$, so that we can integrate not from $-\infty$, but from $-\mu$. This is essential to avoid divergences - because (3.30) is not obligatory for the **approximate** $G_{\mathrm{R/A}}$ which we always deal with. Surely we can do this also in the interaction case (like in sec. 10.2), but with the lower limit slightly changed (due to the change of the lowest energy level). To avoid this, we can set a sort of a "boundary condition": zero of energy = place of the lowest energy level. How this would be compatible with the perturbation theory - a nice question. 

Note also that (8.27) coincides with ([21]4.3). The difference is due to the different normalization of wave functions: I use the normalization without summation over spins (because we have no spin effects in the whole of this text), while in [21] spins are considered.

Part IV

Interaction effects

Chapter 9

Keldysh technique

Correspondence between [5] and [21]: ([5]6.23)=([21]2.10), ([5]6.25)=([21]2.14), ([5]6.32)=([21]2.16). See also NLσM: [89], [17], 9810191, [26]. See also cond-mat/0506752.

9.1 The Keldysh contour

Let us study the case of a quasi equilibrium state when an external fixed parameter is the energy distribution function [38]. More precisely, we require that the unperturbed DM commutes with the unperturbed Hamiltonian.

Let us introduce a new time variable, which we call "Keldysh time". While usual time is described by one continous real variable, Keldysh time has in addition a discreete index, which can have two values: f =forward and b =backward. In other words, while usual time is defined on $\mathbb{R} \equiv (-\infty, +\infty)$, Keldysh time is defined on $(t_0, t_1) \otimes \{f, b\}$, where about t_0 we speak later. There is a trivial projection rule from $(t_0, t_1) \otimes \{f, b\}$ to (t_0, t_1), which consists in removal of the discrete part of the Keldysh time variable. Below we use only this type of projection.

In the equilibrium $T = 0$ diagrammatic technique [5], usual time-ordering T is used. This is just a trivial ordering on $\mathbb{R}$. Let us now define ordering on $(t_0, t_1) \otimes \{f, b\}$, and call it T_C.

A pictorial (=наглядный,anschaulich) way to introduce T_C is using so called "Keldysh contour" depicted in fig. 9.1. Let us define T_C for the case of two operators, thus having two times to order[1]. The difference between usual $T = 0$-technique ordering T and T_c is that while T depends only on the relation between the two times, T_c has also a selected point t_0. This is equavalent to say that in reality we have 3 times to order: two times are "external arguments" of the T_c-operation, while the third time is fixed. This third time corresponds to some point t_0 on a real time axis.

Some (half-baked) notes on the derivation on pp. [21]326-327: In ([21]2.12), there are three different Keldysh time orderings (since they have different[2] "point of return" t_0). Two "inner" orderings get their t_0 from the main (i.e., external) one using projection operation (defined above). The t_0 time for the main (i.e., external) ordering is given by $\max(t_1, t_1')$; however, I suspect that nothing chages if I just set $t_0 = +\infty$. In order to understand this, it would be useful to write the analog of ([21]2.10) for $u^\dagger$.

The definition of the one- and two particle Green functions in Keldysh technique is analogous to that in the $T = 0$ technique ([5]7.1,10.12); only now instead of the usual time ordering, Keldysh ordering is used:

$$G_C(x, x') \equiv -i\langle T_c \left[\psi(x)\psi^\dagger(x')\right]\rangle, \quad G_C^{\text{II}}(x_1, x_2; x_3, x_4) \equiv \langle T_c \left[\psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)\right]\rangle, \tag{9.1}$$

[1] I believe that also for T_C the analogy with a usual sorting is valid: having defined the sorting rule for 2 elements we are able to generalize this it to the arbitrary number of elements.

[2] I would prefer to call t_0 репер(=Richtpunkt, Festpunkt) if I knew how it is translated into English.

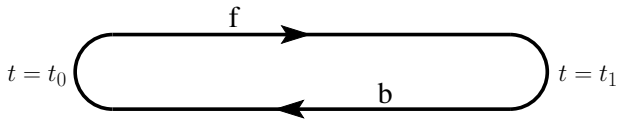


Figure 9.1: The *finite* Keldysh contour: $t_0 \le t \le t_1$. See also [9810191] and Sec. 4.1. See restrictions in (13.28).

where now all time arguments are "Keldysh times" defined on p. 78. This immediately suggests mapping of G_C onto 2×2 matrix, and $G_C^{\rm II}$ from (9.1) — onto 4-(rank) matrix[3].

9.2 One-particle quantities

One-particle quantities like, charge current, are given by G_C (or by its 2×2 matrix mapping). One immediately notes that we are almost always interested in the off-diagonal elements of this matrix, which gives the density matrix – like quantities (9.4). Then using (13.24) we transform our 2×2 matrix[4]

$$\begin{aligned}
G_C \leftrightarrow \begin{pmatrix} G_{\rm ff} & G_{\rm fb} \\ G_{\rm bf} & G_{\rm bb} \end{pmatrix} &= \begin{pmatrix} -i\langle T[\psi(x)\psi^\dagger(x')]\rangle & -i\eta\langle\psi^\dagger(x')\psi(x)\rangle \\ -i\langle\psi(x)\psi^\dagger(x')\rangle & -i\langle\tilde T[\psi(x)\psi^\dagger(x')]\rangle \end{pmatrix} = \\
&= \frac{1}{2}\begin{pmatrix} G_{\rm R}^{(+)} + G_{\rm R}^{(-)} - \eta\left(G_{\rm A}^{(+)} - G_{\rm A}^{(-)}\right) & \eta\left[G_{\rm R}^{(+)} - G_{\rm R}^{(-)} - \left(G_{\rm A}^{(+)} - G_{\rm A}^{(-)}\right)\right] \\ G_{\rm R}^{(+)} + G_{\rm R}^{(-)} - \left(G_{\rm A}^{(+)} + G_{\rm A}^{(-)}\right) & -\left(G_{\rm A}^{(+)} + G_{\rm A}^{(-)}\right) + \eta\left(G_{\rm R}^{(+)} - G_{\rm R}^{(-)}\right) \end{pmatrix} = \\
&= \frac{1}{2}\begin{pmatrix} G_{\rm K} + (G_{\rm R} + G_{\rm A}) & G_{\rm K} - (G_{\rm R} - G_{\rm A}) \\ G_{\rm K} + (G_{\rm R} - G_{\rm A}) & G_{\rm K} - (G_{\rm R} + G_{\rm A}) \end{pmatrix} \equiv \Gamma, \quad L\sigma_3\Gamma L^{-1} = \begin{pmatrix} G_{\rm R} & G_{\rm K} \\ 0 & G_{\rm A} \end{pmatrix}, \quad L = \frac{\sigma_0 - i\sigma_2}{\sqrt{2}},
\end{aligned} \tag{9.2}$$

where T and $\tilde T$ stand for normal and inverse time ordering; all Green functions have (x,x') arguments. It is more convenient to work in the representation where Green function is given by the upper triangular matrix $G = \begin{pmatrix} G_{\rm R} & G_{\rm K} \\ 0 & G_{\rm A} \end{pmatrix}$.

See `deriveKeldysh.nb` for the details of the corresponding transformation.

As far as I remember, γ-matrices ([21]2.44,45) are exchanged. The correct ones are written in `diagrams.m` This corresponds to the usual "from left to right" way of thinking and drawing diagrams (which is opposite to that used in [21], see fig. [21]6.)

With the notations (13.19) we have from ([21]2.22) (in the following, upper signs stand for bosons and the lower ones - for fermions):

$$G_{\rm K}(\lambda,\lambda') = -i\langle[\psi(\lambda),\psi^\dagger(\lambda')]_\pm\rangle = G_{\rm R}^{(\pm)}(\lambda,\lambda') - G_{\rm A}^{(\pm)}(\lambda,\lambda'), \tag{9.3}$$

Note that $G_{\rm R/A}^{(\mp)}$ are the usual retarded and advanced Green functions (see [5]).

In order to calculate one-particle quantities (like e.g., current), we need to calculate expectation values of $G_{\rm fb} \propto \langle\psi^\dagger(x')\psi(x)\rangle$ [which has the same order of operators, as the one-particle density matrix]. One obtains that for fermions [in (non?)equilibrium but stationary state]

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-iEt/\hbar}\langle\hat\psi(\lambda,t)\hat\psi^\dagger(\lambda',0)\rangle {\rm d}t &= \frac{i}{2}\left[G_{\rm R} - G_{\rm A} + G_{\rm K}\right](\lambda,\lambda';E), \\
\int_{-\infty}^{\infty} e^{-iEt'/\hbar}\langle\hat\psi^\dagger(\lambda',t')\hat\psi(\lambda,0)\rangle {\rm d}t' &= \frac{i}{2}\left[G_{\rm R} - G_{\rm A} - G_{\rm K}\right](\lambda,\lambda';E). \\
\langle\hat\psi^\dagger(\lambda',t')\hat\psi(\lambda,t)\rangle &= \frac{i}{2}\left[G_{\rm R} - G_{\rm A} - G_{\rm K}\right](\lambda,t;\lambda',t').
\end{aligned} \tag{9.4}$$

See [20] for the derivation of the Kubo conductivity formula... See also Yudson's proof from my thesis [88], but it seems to me that it is valid only for the case of constant $\vec A$.

From ([19]2.8) and ([19]2.10) it follows that[5] in equilibrium $G^{(+)} = G^{(-)}\coth\frac{E}{2T}$, and we obtain that

$$G_{\rm K}^E = h_E(G_{\rm R}^E - G_{\rm A}^E) \tag{9.5}$$

with h_E from (9.9). In equilibrium from (9.5) and from the usual charge current formula [20] one obtains (8.27). In our article [91] (see sec.11) we argue (but do not prove definitely) that $G_{\rm R/A}$ contain the information only about diagonal MEs[6]. Thus any physical quantity defined using $G_{\rm R/A}$ (like tunnel density of states) can be written in an equilibrium form (see (8.27) as an example):

$$O = \int {\rm d}E \frac{1}{1+e^{E/T}} O(E). \tag{9.6}$$

[3] I mean, a matrix with four indices; each index has two possible values: f or b. In the zeroth order of the perturbation theory, two diagonal elements $G_{\rm ffff}^{\rm II}$ and $G_{\rm bbbb}^{\rm II}$ will be expressed in terms of $G_{\rm ff}$ and $G_{\rm bb}$ [according to ([5]10.14)].

[4] My notations $G_{\rm ff}$, $G_{\rm fb}$, etc correspond to G_{++}, G_{+-}, etc in §[12]92. In the western literature, other notations are used, see §[90]2.1: $G_{\rm fb} \equiv G^<$.

[5] Out of equilibrium, e.g., see p. 26 in `cond-mat/0506617`.

[6] One can think about an analogy: in the 2×2 matrix Green function they stand on diagonal, so they know only about diagonal elements of the density matrix.

I think that in equilibrium (9.6) holds for *any* one-particle physical quantity.

Let us consider an equilibrium of non-interacting particles in some external potential, so that λ is a conserving quantity. From (13.29) and (13.30) we get:

$$G_{\mathrm{R}}^{(\pm)(0)}(\lambda,E) = \frac{h_{\mathrm{B}/F}(\xi_\lambda)}{E-\xi_\lambda+i\epsilon}, \quad G_{\mathrm{A}}^{(\pm)(0)}(\lambda,E) = \frac{h_{\mathrm{B}/F}(\xi_\lambda)}{E-\xi_\lambda-i\epsilon}, \quad \epsilon=+0, \tag{9.7}$$

$$G_{\mathrm{K}}^{(0)}(\lambda,E) = h_{\mathrm{B}/F}(\xi_\lambda)\left(G_{\mathrm{R}}^{(\mp)(0)}(\lambda,E)-G_{\mathrm{A}}^{(\mp)(0)}(\lambda,E)\right), \tag{9.8}$$

$$h_{\mathrm{B}}(E) = \begin{cases} 0, & E<0 \\ \coth\frac{E}{2T}, & E\geq 0 \end{cases}, \quad h_{\mathrm{F}}(E) = \begin{cases} 0, & E<-E_{\mathrm{F}} \\ \tanh\frac{E}{2T}, & E\geq -E_{\mathrm{F}} \end{cases}. \tag{9.9}$$

In the bosonic case ξ_λ is the energy of a boson; In the fermionic case ξ_λ is the energy of a fermion measured from the Fermi energy.

We have from (9.8):

$$G_{\mathrm{K}}^{(0)}(\lambda,E) = -2\pi i \times h_{\mathrm{B}/F}(\xi_\lambda) \times \delta(E-\xi_\lambda),$$

so that (in accordance with ([21]3.21) , ([21]2.66) and (9.5))

$$G_{\mathrm{K}}^{(0)}(\lambda,E) = h_{\mathrm{B}/F}(E)\left(G_{\mathrm{R}}^{(\mp)(0)}(\lambda,E)-G_{\mathrm{A}}^{(\mp)(0)}(\lambda,E)\right). \tag{9.10}$$

Let us now focus on the case of electrons. We are used to work with their Green functions averaged over the interaction with randomly placed impurities. After the averaging we get

$$\langle G_{\mathrm{K}}(\vec{p},E)\rangle = h_F(E)\left(\langle G_{\mathrm{R}}(\vec{p},E)\rangle - \langle G_{\mathrm{A}}(\vec{p},E)\rangle\right), \tag{9.11}$$

where $G_{\mathrm{R/A}}$ are given by (1.9).

Consider a special type non-equilibrium system such that (9.5) holds with some (non-equilibrium) h_E. Suppose a physical quantity is given by a time-independent operator:

$$f(\vec{r},t) = \lim_{\vec{r}'\to\vec{r}} \hat{f}(\vec{r},\vec{r}')\langle\psi^\dagger(\vec{r}',t)\psi(\vec{r},t)\rangle.$$

Then from (9.4) we get general expression for some (almost arbitrary) physical quantity W:

$$W(\vec{r}) = \int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\frac{1-h_E}{2}W(\vec{r},E), \quad W(\vec{r},E) = \int\frac{\mathrm{d}^d p\mathrm{d}^d p'}{(2\pi)^{2d}}e^{i\vec{r}(\vec{p}-\vec{p}')}\hat{W}(\vec{p},\vec{p}')(G_{\mathrm{R}}-G_{\mathrm{A}})(\vec{p},\vec{p}').$$

In particular, this holds for the current [see (8.27)] and the particle density. Without the interaction $G_{\mathrm{R}}-G_{\mathrm{A}}$ is the density of states, in the general case it is called tunnel density of states and manifests strong decay near the Fermi-energy level under the influence of even small interaction (ZBA).

A common application of Keldysh technique is a steady non-equilibrium state of a system with the interaction. It can be introduced via the ansatz for $G_{\mathrm{K}}^{(0)} = h_E(G_{\mathrm{R}}^{(0)}-G_{\mathrm{A}}^{(0)})$ with the appropriate (non-equilibrium) h_E. Some one can object that in this way we do not describe the presumed state of the system completely, because we do not insert off-diagonal elements of the density matrix into it. However, once diagonal elements in the zeroth-order approximation are fixed, the off-diagonal ones are completely defined by the interaction. Thus one can say that non-equilibrium steady state can be *completely* characterized by its energy distribution function f_E.

I think the formulas in section 3.1 are true also for G_{c_K} from ([21]2.18). Also they should be valid for $\hat{G}$ if one substitutes $\vec{A}$ by $\tau^0\vec{A}$ - see ([21]2.41).

So, we get from ([21]4.2), (3.12) and (9.8) in accordance with (3.20) and (3.21): STOP to be revised because ([21]4.3) is incorrect!

$$\begin{aligned}\vec{j}(q)+\frac{Ne^2}{m}\vec{A}(q) = \frac{ie^2}{4m^2}\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\big[(h_0(E)-h_0(E-\omega))\,G_{\mathrm{R}}(p)G_{\mathrm{A}}(p-q)+G_{\mathrm{R}}(p-q)G_{\mathrm{R}}(p)h_0(E)- \\ -G_{\mathrm{A}}(p-q)G_{\mathrm{A}}(p)h_0(E)\big](2\vec{p}-\vec{q})\vec{A}(q)(2\vec{p}-\vec{q}),\end{aligned} \tag{9.12}$$

The lack of "ones" in ([21]4.7) is apparent; they can be added just like in sec. 3.2: because their poles are on one side of the complex E - plane, they give zero contribution into the integral by E.

Note that formula (8.27) used in section 8.12 can be easily obtained from Keldysh technique; actually $G_{\mathrm{R}}-G_{\mathrm{A}}$ comes from the zeroth - order Keldysh component of $\hat{G}$, see (9.8) and...

9.3 Two-particle quantities, e.g. current correlator

A two-particle quantity can be calculated out of the following two particle Green function:

$$\langle\hat{\psi}^\dagger(\vec{r}_1',\sigma_1',t_1)\hat{\psi}(\vec{r}_1,\sigma_1,t_1)\hat{\psi}^\dagger(\vec{r}_2',\sigma_2',t_2)\hat{\psi}(\vec{r}_2,\sigma_2,t_2)\rangle \tag{9.13}$$

At first I must express the correlator in terms of G_C^{II} defined in (9.1). The required [i.e., the same as in (9.13)] sequence of operators can be achieved in several elements of the 4-matrix (p. 79) G_C^{II}. However, the best element to look at is $G_{\text{fbfb}}^{\text{II}}$:

$$-G_{\text{fbfb}}^{\text{II}}(x_2,x_1;x_4,x_3)=\langle\Big(T\Big[\psi(x_1)\psi^\dagger(x_3)\Big]\Big)\Big(T\Big[\psi(x_2)\psi^\dagger(x_4)\Big]\Big)\rangle. \tag{9.14}$$

We get (9.13) from (9.14) for $t_3=t_1+0, t_4=t_2+0$, $\vec{r}_3=\vec{r}_1$, $'s_3=s_1'$, $\vec{r}_4=\vec{r}_2'$, $s_4=s_2'$.

Now let us derive the expression for $G_{\text{fbfb}}^{\text{II}}$ in the external field, but without interaction. [It is now oportune to read [5] between ([5]10.12) and ([5]10.14).] The absence of interaction must simplify perturbation theory for G_C^{II} down to the perturbation theory for G_C:

$$\begin{aligned}G_C^{\text{II}}(x_2,x_1;x_4,x_3)&=G_C(x_2,x_4)G_C(x_1,x_3)-G_C(x_2,x_3)G_C(x_1,x_4),\\ G_{\text{fbfb}}^{\text{II}}(x_2,x_1;x_4,x_3)&=G_{\text{ff}}(x_2,x_4)G_{\text{bb}}(x_1,x_3)-G_{\text{fb}}(x_2,x_3)G_{\text{bf}}(x_1,x_4),\end{aligned} \tag{9.15}$$

From the diagrammatical point of view, the first term in (9.15) is similar to the UCF-type current-current correlator from [88] and sec. **??**. However, the components involved are $G_{\text{ff}}(x_1,x_3)$ and $G_{\text{bb}}(x_2,x_4)$, instead of $G_{\text{fb}}(x_1,x_3)$ and $G_{\text{fb}}(x_2,x_4)$ [which are implied in sec. **??**]. But the Keldysh components are the same in both cases, and thus I suspect that the first term [in the second order perturbation theory in $\vec{A}$] in (9.15) gives us exactly the UCF-type current-current correlator from sec. **??**. One can note that this first term will be time independent. Without disorder averaging, it gives just the square of a curent expectation value[7].

On the contrary to the first term, the second one will depend on difference of times of two current operators[8]. Due to the fact, that an unperturbed Green's function gives zero charge current, only second (exchange) terms in (9.15) survive in the linear responce:

$$\begin{aligned}\delta G_C^{\text{II}}(x_1,x_2;x_3,x_4)&=-\delta G_C(x_1,x_4)G_C(x_2,x_3)-G_C(x_1,x_4)\delta G_C(x_2,x_3),\\ \delta G_{\text{fbfb}}^{\text{II}}(x_1,x_2;x_3,x_4)&=-\delta G_{\text{fb}}(x_1,x_4)G_{\text{bf}}(x_2,x_3)-G_{\text{fb}}(x_1,x_4)\delta G_{\text{bf}}(x_2,x_3),\end{aligned} \tag{9.16}$$

From (9.2) we see that we need perturbation theory expressions separately for δG_{K} and for $\delta(G_{\text{R}}-G_{\text{A}})$, which I already have, see (3.38) and (3.39).

9.4 Kinetic equation

See also: arXiv/0710.3222. See sec. 3.8.

See also p. 30-32 from cond-mat/0210125 about Eilenberger & Usadel equations; About usadel equation, see pp. [21]352353. About Eilenberger with Keldysh & SOI see cond-mat/0601525.

Suppose we have right- and left-hand Dyson equation for the 2x2 matrix Green function with some self-energy σ

$$G=G^{(0)}+G^{(0)}\otimes\sigma\otimes G,\quad G=G^{(0)}+G\otimes\sigma\otimes G^{(0)},\quad (G^{(0)-1}-\sigma)\otimes G=\hat{1},\quad G\otimes(G^{(0)-1}-\sigma)=\hat{1}$$

or

$$(G^{(0)-1}-\sigma)\otimes G=\hat{1},\quad G\otimes(G^{(0)-1}-\sigma)=\hat{1}.$$

Then we subtract one equation from another. The only non-zero component of the resulting matrix equation is the Keldysh one.

Now let us explore several different cases of σ: (i) when we have no interaction but just averaging over the disorder, (ii) when we have a metalic strip subjected to the voltage, and (iii) when we have interaction. See dephasing.pdf for more information.

---

[7] Since now we have two types of averaging, there are two different possibilities of defining a central moment. ask DL: Which is the correct one? The disorder averaging is special: we can do it only at after quantum one(s).

[8] It can not depend on both times, but only on difference, because we are considering steady non-equilibrium state.

In Korea I asked Halperin and EM about cond-mat/0505535. EM told me that it must be possible to get the effect of side jump[9] from the kinetic equation, only one has to understand, what is the important set of diagrams. E.g., weak localization one catches with the kinetic equation on p. [21]344.

The first non-vanishing contribution to the QKE is given by the second order (in interaction) diagrams, see p. [12]487.

9.5 Функциональные интегралы

Эту секцию надо будет объединить с секцией 4.

9.5.1 Введение в технику Келдыша

Обозначим: $|\vec{p}\rangle$ = элемент линейного пространства функций, собственная функция (*без указания представления*) оператора импульса $\hat{\vec{p}}$, соответствующая с.ч. $\vec{p}$. $\langle\lambda|$ = линейная форма, или элемент сопряжённого линейного пространства (вспомним начало курса линейной алгебры). Соответственно, $\langle\lambda|\vec{p}\rangle \in \mathbb{C}$ есть скалярное произведение. Разумеется, численное значение $\langle\lambda|\vec{p}\rangle$ зависит от λ и $\vec{p}$. Напр., $\langle\vec{r}|\vec{p}\rangle$ совпадает со значением с.ф. импульса в точке $\vec{r}$: $\langle\vec{r}|\vec{p}\rangle = \exp[i\vec{p}\vec{r}]/\sqrt{V}$.

В случае, если гамильтониан квадратично зависит от импульса, матричный элемент оператора эволюции может быть записан в виде (см. стр. [51]61):

$$\left\langle x_f \left| \exp\left[-\frac{i}{\hbar}\hat{H}(\hat{\vec{x}},\hat{\vec{p}})(t_f-t_i)\right] \right| x_i \right\rangle = \int D[x(t)] \exp\left\{-\frac{i}{\hbar}S[x(t)]\right\},$$
$$S[x(t)] = \int_{t_i}^{t_f} \mathrm{d}t L[x(t)], \quad L[x(t)] = \frac{m\dot{x}^2}{2} - V(x). \tag{9.17}$$

В более общем случае ([51]2.48)

$$\left\langle x_f \left| \exp\left[-\frac{i}{\hbar}\hat{H}(\hat{\vec{x}},\hat{\vec{p}})(t_f-t_i)\right] \right| x_i \right\rangle = \int D[x(t)p(t)] \exp\left\{-\frac{i}{\hbar}\left[p\dot{x} - H(p,x)\right]\right\}. \tag{9.18}$$

Теперь давайте вместо координат (т.е. с.ч. операторов координат) использовать пространство с.ч. операторов уничтожения. Например, это могут быть операторы уничтожения в координатном или импульсном пространстве. Тогда получаем[10]

$$\left\langle \varphi_f \left| \exp\left[-\frac{i}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})(t_f-t_i)\right] \right| \varphi_i \right\rangle = \int D[\varphi^*(t)\varphi(t)] \exp\left\{-\frac{i}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t \sum_\alpha \left[i\hbar\varphi_\alpha^*\dot{\varphi}_\alpha - H(\varphi_\alpha^*,\varphi_\alpha)\right]\right\}, \tag{9.19}$$

$$\sum_{\alpha=1}^{2N+1}\left[i\hbar\varphi_\alpha^*\dot{\varphi}_\alpha - H(\varphi_\alpha^*,\varphi_\alpha)\right] = \sum_{\alpha=1}^{N}\left[i\hbar\varphi_\alpha^*\dot{\varphi}_\alpha - H(\varphi_\alpha^*,\varphi_\alpha)\right] + \sum_{\alpha=N+1}^{2N+1}\left[i\hbar\varphi_\alpha^*\dot{\varphi}_\alpha - H(\varphi_\alpha^*,\varphi_\alpha)\right]. \tag{9.20}$$

Это очень похоже на Гамильтонову форму записи (9.18) эволюционного оператора. Координате соответствует φ, а импульсу – φ^*. Это естественно, если вспомнить, что соотношения коммутации между операторами $\hat{\vec{r}}$ и $\hat{\vec{p}}$ аналогичны соотношениям коммутации между $\hat{a}$ и $\hat{a}^\dagger$, и вообще, напр., в [68] в некоторых доказательствах используется подстановка $\hat{a}^\dagger \to \frac{\partial}{\partial\hat{a}}$.

Матрица ([26]5) учитывает доп. эксп. множитель в ([51]2.62a), а также обращение времени в центре временного интервала.[11] Элемент в верхнем правом углу соединяет начало контура с его концом, подчёркивая, что интегрирование проходит по *замкнутому* контуру. В этом элементе содержится $\rho(\omega_0)$, т.к. оператор ρ_0 в ([26]2)

[9] See p. **??** for my new ideas about side jump.

[10] В дискретных обозначениях φ_α есть значение φ в момент времени t_α. Следует однако же помнить, что $\forall\alpha$ φ_α – бесконечномерный (и не обязательно счётный) массив комплексных чисел; разные числа соответствуют различным операторам уничтожения, напр., $\hat{\psi}_{\vec{r}_1}(t_\alpha)$ и $\hat{\psi}_{\vec{r}_2}(t_\alpha)$.

[11] Замечу, что матричные элементы в верхних правых углах матриц ([26]5) и [более поздней] ([50]10) *отличаются*. Странно, не правда ли?

относится именно к начальному моменту времени $t_i \equiv t_1 \equiv t_{2N+1} \equiv t_f$. Итак, переписываем действие ([26]4) в явном виде: $S = \lim_{N\to\infty} S_N$, где

$$i\delta_t S_N[\varphi^*,\varphi] = \varphi_1^*\left[(1-\delta_t i\omega_0)\,\varphi_{2N+1}\rho(\omega_0) - \varphi_1\right] + \sum_{\alpha=1}^{N}\varphi_{\alpha+1}^*\left[(1-\delta_t i\omega_0)\,\varphi_\alpha - \varphi_{\alpha+1}\right] + \\ + \sum_{\alpha=N+1}^{2N+1}\varphi_{\alpha+1}^*\left[(1+\delta_t i\omega_0)\,\varphi_\alpha - \varphi_{\alpha+1}\right]. \tag{9.21}$$

При $\alpha = 1\ldots N$ выражения величина $t_{\alpha+1} - t_\alpha = \delta_t > 0$, в то время как $\forall\alpha = N\ldots 2N+1$ $t_{\alpha+1} - t_\alpha = -\delta_t < 0$, что привело к изменению знака перед δ_t в последнем слагаемом.

Следующий шаг: вместо $(\varphi_1\ldots\varphi_{2N+1})$ вводим двухкомпонентное поле $(\Phi_1\ldots\Phi_N)$. А именно $\forall i = 1\ldots N$ величина Φ_i имеет две компоненты, обозначаемые индексами f и b: $\Phi_{if} \equiv \varphi_i$ и $\Phi_{ib} \equiv \varphi_{N+1+i}$. STOP Получается, что из расмотрения выпадает "поворотный" элемент φ_{N+1}. Аналогично переходим от φ^* к Φ^*. В новых обозначениях

$$\lim_{N\to\infty}\delta_t\sum_{\alpha=1}^{N}\varphi_{\alpha+1}^*\left[\frac{\varphi_\alpha-\varphi_{\alpha+1}}{\delta_t} - i\omega\varphi_\alpha\right] = \int_{-\infty}^{\infty}\Phi_f^*\left(-\dot\Phi_f - i\omega\Phi_f\right)\mathrm{d}t, \\ \lim_{N\to\infty}\delta_t\sum_{\alpha=N+1}^{2N+1}\varphi_{\alpha+1}^*\left[\frac{\varphi_\alpha-\varphi_{\alpha+1}}{\delta_t} + i\omega\varphi_\alpha\right] = \int_{-\infty}^{\infty}\Phi_b^*\left(\dot\Phi_b + i\omega\Phi_b\right)\mathrm{d}t, \tag{9.22}$$

так что в непрерывных обозначениях (9.21) м.б. переписано следующим образом:

$$iS[\Phi^*,\Phi] = \Phi^*\hat G_0^{-1}\Phi, \quad \Phi^* = (\Phi_f^*,\Phi_b^*), \quad \Phi = \begin{pmatrix}\Phi_f\\ \Phi_b\end{pmatrix}, \quad \hat G_0^{-1} \equiv \clubsuit\begin{pmatrix} i\frac{\partial}{\partial t} + i\omega & 0\\ 0 & -i\frac{\partial}{\partial t} - i\omega\end{pmatrix}. \tag{9.23}$$

Непрерывная форма записи (9.23) содержит меньше информации, чем (9.21), причём эта недостающая информация важна. Нам же хочется забыть про дискретную форму записи и впредь пользоваться только непрерывной формой. Я привык к записи ГФ в форме (9.1), а теперь мне придётся привыкать к форме ([26]8). Следует понимать, что средние $\langle\ldots\rangle$ в (9.1) и ([26]8) – разные: (9.1) усредняется согласно (13.20) и (13.21), в то время как

$$\langle\Phi_i\Phi_j^*\rangle = \int D[\Phi^*(t)\Phi(t)]\Phi_i\Phi_j^*\exp\left\{-\frac{i}{\hbar}S[\Phi^*,\Phi]\right\}, \quad i,j\in\{f,b\}. \tag{9.24}$$

Для перехода между представлением чисел заполнения и когерентными состояниями теоретически можно использовать матр. элемент $\langle n|\varphi\rangle$ ([51]1.114,1.139). Строгого доказательства того, почему эти два средних совпадают, я не знаю. В ([51]2.67b) соответствующий переход делается тихонечко, без шума и пыли, и вообще – не заостряя внимание.

Нам хочется забыть навсегда о дискретных обозначениях. Для этого мы разделяем поля φ и φ^* на "квантовые" и "классические" согласно ([26]10). Если мы теперь будем иметь в ввиду, что величина $\left[G^{-1}\right]^{-1}$ в ([50]32) не равна нулю (но есть беск. малое ϵ), то матрица G у нас теперь обратима, и мы можем отныне пользоваться непрерывными обозначениями, лишь в минуту душевной невзгоды вспоминая про дискретизацию. Наше *квадратичное* действие однозначно определяется средними значениями ([26]11). Таким образом, нам становится понятным ([26]19).

Дальше танцуем от начала секции [50]6. Преобразование Хаббарда-Стратоновича фактически имеет смысл Фурье-преобразования. Действительно, нам не нравится U- представление тем, что $U(r)$ в нём – очень сильно осциллирующая функция. И тогда мы переходим в (сопряжённое ему) Q- представление.[12]

9.5.2 Вспоминаем о беспорядке

В системе с беспорядком действие записывается в форме ([52]4):

$$S[\bar\psi,\psi] = \oint\mathrm{d}t\int\mathrm{d}^d r\bar\psi(\vec r,t)\left[G_0^{-1} - U_{\mathrm{dis}}(\vec r)\right]\psi(\vec r,t), \quad G_0^{-1} \equiv i\frac{\partial}{\partial t} + \frac{\nabla_r^2}{2m}, \quad \bar\psi \equiv \{\varphi_\alpha^*\}, \quad \psi \equiv \{\varphi_\alpha\}.$$

[12] Интересно проверить: приводится ли действие в исходное состояние двукратным преобр. Хаббарда-Стратоновича?

Чтобы забыть про сложную структуру матрицы ([26]5), давайте введём вместо ψ двухкомпонентное фермионное поле Ψ (ну и заодно – усреднение по беспорядку) так, что ([52]7a,b)

$$Z = \tilde{\mathcal{N}} \int \mathcal{D}[\overline{\Psi}\Psi] \int \mathcal{D}U_{dis} \exp\left\{-\pi\nu\tau \int d\mathbf{r} U_{dis}^2(\mathbf{r})\right\} \exp\left\{i\,\mathrm{Sp}\left[\overline{\Psi}\left(\hat{G}_0^{-1} - U_{dis}\sigma_3\right)\Psi\right]\right\},$$

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \hat{G}_0^{-1} \equiv \begin{pmatrix} i\frac{\partial}{\partial t} + \frac{\nabla_r^2}{2m} & 0 \\ 0 & -i\frac{\partial}{\partial t} - \frac{\nabla_r^2}{2m}\end{pmatrix}.$$

Берём (квадратичный) интеграл по переменной U_{dis} (для 2D случая):

$$\begin{aligned}
&\int D[U_{dis}] \exp\left\{-\pi\nu\tau \int \mathrm{d}^2 r U^2(\vec{r}) - i\,\mathrm{Sp}\left[\Psi U\sigma_3\Psi\right]\right\} = \\
&= \lim_{N\to\infty} \prod_{i,j=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}U_{ij} \exp\left[-\pi\nu\tau\delta_r^2 U_{ij}^2 - iU_{ij}\left(\psi_{fij}\bar{\psi}_{fij} - \psi_{bij}\bar{\psi}_{bij}\right)\right] = \\
&= \lim_{N\to\infty} \prod_{i,j=1}^{N} \left(\frac{1}{\nu\tau\delta_r^2}\right)^{N^2/2} \exp\left[-\frac{\left(\psi_{fij}\bar{\psi}_{fij} - \psi_{bij}\bar{\psi}_{bij}\right)^2}{\pi\nu\tau\delta_r^2}\right].
\end{aligned} \tag{9.25}$$

После деления на нормализационный множитель

$$\int D[U_{dis}] \exp\left[-\pi\nu\tau \int \mathrm{d}^2 r U^2(\vec{r})\right] = \lim_{N\to\infty} \prod_{i,j=1}^{N} \left(\frac{1}{\nu\tau\delta_r^2}\right)^{N/2}$$

результат интегрирования по U становится конечным ([50]154). После перехода к квантовым и классическим полям, см. секцию [50]5.3, мы видим, что к новым Грассмановым переменным потенциал беспорядка цепляется через единичную матрицу, на через σ_3, как в (154[50]) – через σ_0. Т.к. мы имеем дело с Грассмановыми переменными, $\left(\psi_{fij}\bar{\psi}_{fij}\right)^2 = 0$ и $\left(\psi_{bij}\bar{\psi}_{bij}\right)^2 = 0$.

Теперь пришло время откатываться назад (т.е. делать Хаббарда-Стратоновича), вводя *матричное* поле Q. Если предположить, что все элементы матрицы Q – абсолютно независимы, то я не представляю, как можно было бы посчитать правую часть ([50]155). Совсем другое дело, если предположить, что матрица Q – эрмитова [см. между ([52]11b) и ([52]12)]. То есть

$$\mathrm{Sp}\, Q^2 = \sum_{i,j\geq 1} Q_{ij}Q_{ji} = \sum_{i,j\geq 1}(a_{ij}^2 + b_{ij}^2), \quad Q_{ij} \equiv a_{ij} + ib_{ij}, \quad a_{ij}, b_{ij} \in \mathbb{R}.$$

Следующая ступенька – понимание параметризации ([50]162): почему G_0^{-1} инвариантна отн. преобразования подобия ([50]162)? Свёртка во временном пространстве соответствует произведению в частотном пространстве. Но из-за временной однородности все матрицы в ([50]162) диагональны в частотном пространстве, а значит, коммутируют. Матрица ([50]161) тоже диагональна в частотном пространстве (т.к. мы интересуемся только стационарными распределениями энергий), но недиагональна в келдышевском – у неё на диагонали стоит $2\mathbb{F}_\epsilon$, см. ([50]161). Поэтому при фиксированной Λ но различных $\mathbb{T}$ в правой части ([50]162) мы получаем различные матрицы Q.

Как известно из метода стационарной фазы, наибольший вклад в наш интеграл даёт такая (или такие) окрестность матричного поля $Q(t,t')$, что функциональная производная от действия ([52]21b) по $Q(t,t')$ равна нулю. Приятно, что это приводит нас к SCBA ([52]23). А вот и первый бонус, который мы получаем из-за использования техники Келдыша вместо Мацубары: результат ([52]23) верен в том числе и для неравновесных функций распределения.

Уравнение SCBA ([52]23) соблюдается при любых $Q(t,t')$ не зависящих от импульса, а также удовлетворяющих условию ([17]30-31) $Q^2 = \mathbb{1}$ (где возведение в квадрат подразумевает также свёртку по времени). STOP Интересно понять, так ли это для нашего сильно анизотропного случая.

9.6 Screening

In this section all the calculations are done for the case of small q. To consider the case of large q one should know the dependence of energy on momentum (which we actually never know). However, one can suppose $E(p) = p^2/(2m)$ and do this (quite long) calculation. It must be just this is the way it is done on pp. [92]158-163.

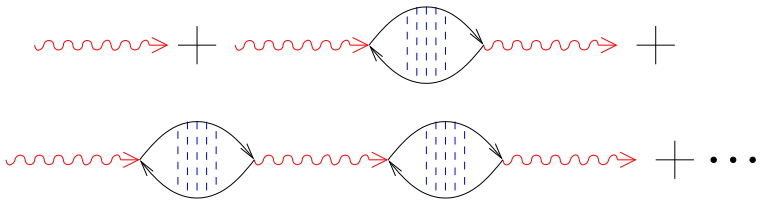


Figure 9.2: Screening in RPA (=приближение хаотических фаз) approximation (wavy lines stand for bare interaction, dashed ones - for the impurity averaging lines composing diffuson). The possibility of connecting of bubbles with "external" green function lines should not be considered here, for it is taken into account when considering diagrams with higher order on interaction. Studying these higher order diagrams, one can detect some other (like RPA or the renormalization in Cooper channel) essential infinite sequence of diagrams. The same is true for the lines connecting different bubbles.

Because of the long-range character of Coulomb interaction, it must not be considered in the first several orders of the perturbation theory. Instead, screening must be taken into account, which is technically done by summation of infinite number of diagrams.

9.6.1 Usual calculation

Let us calculate a bubble from Fig. 9.2:

$$\int \frac{\mathrm{d}^{d+1}p}{(2\pi)^{d+1}} G^{(0)}(p)G^{(0)}(p-q) = i\int_{\left(\substack{p>p_F\\ |\vec{p}-\vec{q}|<p_F}\right)} \frac{\mathrm{d}^d p}{(2\pi)^d}\left\{\frac{1}{\omega+\left(\xi_{\vec{p}}-\xi_{\vec{p}-\vec{q}}\right)-2i\delta} - \frac{1}{\omega-\left(\xi_{\vec{p}}-\xi_{\vec{p}-\vec{q}}\right)+2i\delta}\right\}. \tag{9.26}$$

Let us consider the case of zero-frequency $\omega = 0$ and of small q, so that $\xi_{\vec{p}} - \xi_{\vec{p}-\vec{q}} = \vec{v}_F\vec{q}$. We see that (9.26) reduces to $\frac{2i}{v_F}\int_{\left(\substack{p>p_F\\ |\vec{p}-\vec{q}|<p_F}\right)} \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{\vec{n}\vec{q}}$, which, using the argumentation from [93], p. 198, is equal to:

$$\frac{2i}{v_F}\int_{\left(\substack{p>p_F\\ |\vec{p}-\vec{q}|<p_F}\right)} \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{\vec{n}\vec{q}} = \begin{cases} \frac{i}{\pi v_F}, & \text{in 1D} \\ \frac{im}{2\pi}, & \text{in 2D} \\ \frac{ip_F m}{2\pi^2}, & \text{in 3D} \end{cases}$$

From here we see that screening exists in all dimensions. In 3D and 2D cases we have

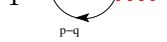
$$U(q) = \frac{4\pi e^2}{q^2+q^{*2}}, \quad U(q) = \frac{2\pi e^2}{q+q^*} \tag{9.27}$$

with different q^* for 2D and 3D (as for the 1D case, Fourier integral for the Coulomb potential diverges).

9.6.2 Calculation by means of Keldysh technique

The renormalized interaction matrix is defined by

$$\tilde{U}(q) = \frac{U(q)}{1 - \Pi(q)U(q)} = \frac{U(q)}{1 - \quad} \tag{9.28}$$



The first and the second diagrams (their matrix kk' components) in fig. 9.2 are respectively equal to $V\delta_{\vec{q},\vec{q}'}U(q)\tau^0_{kk'}$ and

$$-iU(q)U(q')\,\mathrm{Sp}\left[\frac{1}{V^2}\sum_{p,p'}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\gamma^k\hat{G}(\vec{p},\vec{p}';E)\tilde{\gamma}^{k'}\hat{G}(\vec{p}'-\vec{q}',\vec{p}-\vec{q};E-\omega)\right].$$

The polarization thus is given by

$$V\delta_{\vec{q},\vec{q}'}\Pi(q) = -i\langle\mathrm{Sp}\left[\frac{1}{V^2}\sum_{p,p'}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}\gamma^k\hat{G}(\vec{p},\vec{p}';E)\tilde{\gamma}^{k'}\hat{G}(\vec{p}'-\vec{q}',\vec{p}-\vec{q};E-\omega)\right]\rangle. \tag{9.29}$$

From (9.29) and (9.5) it follows that (see (2.8) for D)

$$V\delta_{\vec q,\vec q'}\Pi_{\mathrm R}(q)=-\frac{i}{2}\frac{1}{V^2}\sum_{p,p'}\int_{-\infty}^{\infty}\frac{\mathrm dE}{2\pi}\Big\{D^{\vec p,\vec p';E}_{\vec p-\vec q,\vec p'-\vec q';E-\omega}(h_E-h_{E-\omega})+$$
$$+V^2\delta_{\vec p,\vec p'}\delta_{\vec q,\vec q'}\left[G_{\mathrm R}(p)G_{\mathrm R}(p-q)h_{E-\omega}-G_{\mathrm A}(p)G_{\mathrm A}(p-q)h_E\right]\Big\}, \tag{9.30}$$

$$V\delta_{\vec q,\vec q'}\Pi_{\mathrm A}(q)=-\frac{i}{2}\frac{1}{V^2}\sum_{p,p'}\int_{-\infty}^{\infty}\frac{\mathrm dE}{2\pi}\Big\{-D^{\vec p'-\vec q',\vec p-\vec q;E-\omega}_{\vec p',\vec p;E}(h_E-h_{E-\omega})+$$
$$+V^2\delta_{\vec p,\vec p'}\delta_{\vec q,\vec q'}\left[G_{\mathrm R}(p)G_{\mathrm R}(p-q)h_E-G_{\mathrm A}(p)G_{\mathrm A}(p-q)h_{E-\omega}\right]\Big\}, \tag{9.31}$$

$$V\delta_{\vec q,\vec q'}\Pi_{\mathrm K}(q)=-\frac{i}{2}\frac{1}{V^2}\sum_{p,p'}\int_{-\infty}^{\infty}\frac{\mathrm dE}{2\pi}\Big\{\Big[D^{\vec p,\vec p';E}_{\vec p-\vec q,\vec p'-\vec q';E-\omega}+D^{\vec p'-\vec q',\vec p-\vec q;E-\omega}_{\vec p',\vec p;E}\Big](1-h_Eh_{E-\omega})+$$
$$+V^2\delta_{\vec p,\vec p'}\delta_{\vec q,\vec q'}\left[G_{\mathrm R}(p)G_{\mathrm R}(p-q)+G_{\mathrm A}(p)G_{\mathrm A}(p-q)\right]h_Eh_{E-\omega}\Big\}. \tag{9.32}$$

From the conservation of total number of particles (see VII, 51) one realizes that

$$\text{for } \vec q=0 \text{ and } \forall\omega\ \Pi_{\mathrm{R/A}}(\vec q,\omega)=0. \tag{9.33}$$

From (9.30), (9.31) and (9.32) we realize that **if one ignores effects due to the E – dependence of the density of states or in the equilibrium** (9.5) holds **exactly**[13] for the polarization and hence for the renormalized potential. In the equilibrium case using (13.32) one recovers (9.9).

One can see that[14]

$$\operatorname{Supp}\left[1-h_{E+\omega/2}h_{E-\omega/2}\right]=\operatorname{Supp}\left[h_{E+\omega/2}-h_{E-\omega/2}\right]=\left\{E:|E|\lesssim\max\left(\frac{\tilde T}{2},\frac{\omega}{2}\right)\right\},$$

so that in (9.32) energy integration is limited in the zone where we can use (1.19). It also means that in (9.30), (9.31) and (9.32), we can

- considering non-pole contributions – to substitute $h_E\leftrightarrow h_{E-\omega}$ and $h_Eh_{E-\omega}\leftrightarrow 1$.
- considering pole contributions having $(h_E-h_{E-\omega})$ and $(1-h_Eh_{E-\omega})$ multipliers – to substitute $\frac{1}{V}\sum_{\vec p,\vec n}\to\int\mathrm d\xi\nu(\xi)$ with weak $\nu(\xi)$ dependence (1.19) and to integrate over ξ before integrating over energy.

So we see that (due to $\int_{-\infty}^{\infty}\frac{\mathrm dE}{2\pi}$) non-pole terms give zero contribution to (9.32).

Let us introduce a quantity

$$\tilde\nu(q)=\frac{i}{2}\frac{1}{V}\sum_{\vec p,\vec n}\int_{-\infty}^{\infty}\frac{\mathrm dE}{2\pi}h_E\left[G_{\mathrm R}(p)G_{\mathrm R}(p-q)-G_{\mathrm A}(p)G_{\mathrm A}(p-q)\right]\in\mathbb R,$$

$$\text{If } \nu \text{ is a constant, then}\quad \tilde\nu(q)=\tilde\nu(0)=\nu_0\int\mathrm dEh_E'/2=\nu_0.$$

$$\frac{1}{V^2}\sum_{p,p'}D^{\vec p,\vec p';E}_{\vec p-\vec q,\vec p'-\vec q';E-\omega}=2\pi\nu_0\tilde\tau\left(\frac{1}{1-X(q)}-1\right)V\delta_{\vec q,\vec q'},\quad \frac{1}{V^2}\sum_{p,p'}D^{\vec p'-\vec q',\vec p-\vec q;E-\omega}_{\vec p',\vec p;E}=2\pi\nu_0\tilde\tau\left(\frac{1}{1-X^*(q)}-1\right)V\delta_{\vec q,\vec q'}, \tag{9.34}$$

where $X(q)$ is given by (13.1).

$$\Pi_{\mathrm R}(q)=\Pi_{\mathrm A}^*(q)=-\frac{i\nu_0\tilde\tau}{2}\int\mathrm dE\frac{X(q)(h_E-h_{E-\omega})}{1-X(q)}-\tilde\nu(q),$$
$$\Pi_{\mathrm K}(q)=-\frac{i\nu_0\tilde\tau}{2}\int\mathrm dE\left[\frac{X(q)}{1-X(q)}+\frac{X^*(q)}{1-X^*(q)}\right](1-h_Eh_{E-\omega})\in\mathfrak I. \tag{9.35}$$

[13] The question is if (9.5) holds *always* or only when these conditions do hold. If it represents a general rule, the first variant of our paper [91] is correct. No, this is not true. In [91] from the formal point of view the mistake was that we thought that the functional derivative $\frac{\delta}{\delta D(E')}UV=V\frac{\delta}{\delta D(E')}U$. As for VK, he says (9.5) holds always. And what about other properties: $U_{\mathrm R}^*(\omega)=U_{\mathrm R}(-\omega)=U_{\mathrm A}(\omega)$? Does it hold always?

[14] Подозреваю, что $\max\left(\frac{\tilde T}{2},\frac{\omega}{2}\right)\cong\tilde T_\omega$ -см. также IX, 40.

Note that in general case Π_K is not proportional to $\Pi_R - \Pi_A$, so that (9.5) does not hold.

The simplest form the interaction has for large values of momentum $\sim p_F$. From (13.4) we deduce that in this case

$$U_K = 0, \quad U_R = U_A = \frac{1}{\tilde{\nu}} \equiv \frac{\Lambda}{\nu_0}, \quad \tilde{\nu} \stackrel{df}{=} \lim_{q\to\infty} \tilde{\nu}(q), \quad \Lambda \stackrel{df}{=} \frac{\nu_0}{\tilde{\nu}}. \tag{9.36}$$

What is the order of magnitude of $\nu_0/\tilde{\nu}$? VK: let us substitute in the fine structure constant $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ sppeed of light with the Fermi speed. Then we get approximately 100 times larger quantity $= 1/r_s \sim 0.2 - 0.3$ – for good metals like Cu or Ag, see p. [8]270.

Without taking into account ν_E - dependence, the polarization simplifies substantially:

$$\Pi_R(q) = \Pi_A^*(q) = -\nu_0 \left[1 + i\omega\tilde{\tau}\frac{X(q)}{1 - X(q)}\right], \quad \Pi_K(q) = -2i\nu_0\tilde{\tau}\tilde{T}_\omega \left[\frac{X(q)}{1 - X(q)} + \frac{X^*(q)}{1 - X^*(q)}\right], \tag{9.37}$$

where

$$\begin{aligned} \tilde{T} \equiv &\tilde{T}_0; \quad \tilde{T}_\omega = \tilde{T}_{-\omega} = \frac{1}{4}\int_{-\infty}^{\infty} dE\,(1 - h_E h_{E-\omega}) \equiv \\ \equiv &\int_{-\infty}^{\infty} dE \left(\frac{f_E + f_{E-\omega}}{2} - f_E f_{E-\omega}\right) = \boxed{\text{in equilibrium}} = \frac{\omega}{2}\coth\frac{\omega}{2T} \underset{\omega\to 0}{\longrightarrow} T. \end{aligned} \tag{9.38}$$

In the particular case[15], when a quasi 1D system is subjected to a voltage at $T = 0$,

$$h_E = \frac{x}{L}\,\mathrm{sign}\left[E - \frac{eU}{2}\right] + \left[1 - \frac{x}{L}\right]\mathrm{sign}\left[E + \frac{eU}{2}\right], \quad \tilde{T} = eU\frac{x}{L}\left[1 - \frac{x}{L}\right]. \tag{9.39}$$

In the middle of the sample $x = L/2$ and

$$h_E = \frac{\mathrm{sign}(E - eU/2) + \mathrm{sign}(E + eU/2)}{2}, \quad \tilde{T}_\omega = \begin{cases} (eU + |\omega|)/4, & |\omega| < eU, \\ |\omega|/2, & |\omega| \geq eU. \end{cases} \tag{9.40}$$

For $T \neq 0$, the width of the "transition region" at $|\omega| = eU$ in (9.40) becomes finite (of the order of T) between $\tilde{T}_\omega$ becomes smooth: $\tilde{T}_\omega \in C^2$. After studying two simple cases (9.38) and (9.40), I suspect that, at sufficiently high frequencies, frequency dependence of $\tilde{T}_\omega$ is universal, $\tilde{T}_\omega \approx |\omega|/2$.

From (9.37) for arbitrary $X(q)$ we arrive to

$$\forall\omega \in \mathbb{C} \quad \tilde{U_K} = \frac{2\tilde{T}_\omega}{\omega}\left(\tilde{U_R} - \tilde{U_A}\right); \quad \forall\omega \in \mathbb{R} \quad \tilde{U_R}(q) = \tilde{U_A}^*(q).$$

In 2D $u^{-1}(q) \propto \nu_0 q r_B$; in 3D $u^{-1}(q) \propto \nu_0 (q r_B)^2$. Обычно ряды по q - степенные, а значит, сходящиеся при $q \sim 1/L$; так что в уравнении Дайсона (9.28) можно принебречь $u^{-1}(q)$. В результате $U_{R/A/K}$ перестают зависеть от исходного, неэкранированного потенциала $u^{-1}(q)$. Боря: такая ситуация называется унитарным пределом. for $|\Pi_{R/A}| \gg r_B/L$ the difference $[\tilde{U_R}(\omega) - \tilde{U_A}(\omega)]$ does not depend on the original (unscreened) potential $u(q)$.

In the diffusion approximation

$$\Pi_{R/A}(q) = -\frac{\nu_0 D q^2}{Dq^2 \mp i\omega}, \quad \Pi_K(q) = -\frac{4\nu_0 i D q^2 \tilde{T}_\omega}{D^2 q^4 + \omega^2}. \tag{9.41}$$

One can note that for all values of q diagonal elements in (9.41) give correct poles in ω - complex plane for the renormalized Coulomb interaction defined by (9.28). **In 2D case**, where $U(q) = 2\pi e^2/q$ we have

$$\begin{aligned} \tilde{U_{R/A}}(q) &= \frac{2\pi e^2}{q} - \frac{4\pi^2 e^4 D\nu_0}{Dq(q + 2\pi e^2\nu_0) \mp i\omega} = \frac{1}{\nu_0}\frac{1}{\frac{q}{2\pi\nu_0 e^2} + \frac{Dq^2}{Dq^2 \mp i\omega}}, \\ \tilde{U_K}(q) &= -\frac{16 i\pi^2 e^4 D\nu_0 \tilde{T}_\omega}{D^2 q^2 (q + 2\pi e^2\nu_0)^2 + \omega^2} = -\frac{4\tilde{T}_\omega}{Dq^2} \cdot \frac{i\nu_0}{\left(\frac{q}{2\pi e^2} + \nu_0\right)^2 + \left(\frac{\omega}{2\pi e^2 Dq}\right)^2}. \end{aligned} \tag{9.42}$$

[15] In principle, when a current flows through a metal, quasiclassical distribution function $f(\vec{p},\vec{r})$ of electrons become anisotropic, see (3.47) and p. [48]347. On the other hand, we know [21], that elastic scattering off impurities wants to make it isotropic. So we conclude [PRB**52**4740] that in case of strong impurity scattering, the distribution function becomes isotropic in momentum $f(\vec{p},\vec{r}) \approx f(p,\vec{r}) \equiv f_E(\vec{r})$. Note that Apparently, there is a clever way [cond-mat/0406063] to understand it in Keldysh technique, which I don't know. Better see PRB**64**033301 and PRB**52**4740. My understanding is more primitive, see pp. [94]65-66.

In 2D $\frac{1}{2\pi e^2 \nu_0} \sim \frac{\pi}{e^2 \nu_0} = r_B$ = Bohr radius. Substituting $\omega = 0$ in (9.42), we see that q^* from (9.27) is equal to $\frac{4\pi^2}{r_B} \sim p_F$.

Like in [39], we want to neglect $\frac{q}{2\pi e^2}$ in the denominator of (9.42). We can do it in case

$$\omega \ll Dq/r_B \ll D/r_B^2 \tag{9.43}$$

and obtain[16]

$$\tilde{U_{R/A}}(q) \approx \frac{1}{\nu_0}\left(1 \mp \frac{i\omega}{Dq^2}\right), \quad \tilde{U_K}(q) \approx \frac{-4i\tilde{T}_\omega}{\nu_0 Dq^2}. \tag{9.44}$$

The last inequivalence in (9.43) holds always; as for the first one, we must see if there are any diagrams for which this condition violates. Particularly, it holds in the *diffusion regime*, when $Dq^2 \sim |\omega|$. Note that always due to the screening of the atomic lattice $U(q=0) = 0$, so that we don't have any divergence.

Let us now go **out of the diffusion approximation**: still $q \ll p_F$ but $ql \gg 1$. Then

$$\Pi_{R/A}(q) = -\nu_0\left[1 \pm \frac{i\omega\tau}{lq}\right], \quad \Pi_K(q) = -\frac{4i\nu_0\tau}{lq}\tilde{T}_\omega,$$

so that

$$\tilde{U}_{R/A}(q) = \frac{1}{\nu_0\left(1 \pm \frac{i\omega\tau}{lq}\right)}, \quad \tilde{U}_K(q) = \frac{1}{\nu_0}\frac{4i\tilde{T}_\omega\tau}{lq}\frac{1}{1+\left(\frac{\omega\tau}{lq}\right)^2}. \tag{9.45}$$

From (13.1), (9.38), (9.44), (9.45) and (13.4) one can say that for $Dq \gg |\omega|$ the potential does not depend on q.

In two types of Ambegaokar & Eckern diagrams in [95] large values of q ($\sim p_F$) are important, so that $\epsilon_q > \omega \sim T$. This leads to a simplification: $\tilde{U}(q) \approx U_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If I understood correctly VK, $U_0 \sim \frac{e^2}{\hbar v_F}\frac{1}{2\nu_0}$. STOP Проблемы с размерностью в $\frac{e^2}{\hbar v_F}$: как мы можем преобразовать Кулоны в сантиметры? - см. estimates.nb и uebung3.nb. Such simplified short range potential is used both in [39] and in [95].

[16] Note that (9.44) is correct in arbitrary dimension.

Chapter 10

The steady state

10.1 Calculating mean values of physical quantities

In the steady state[1] the energy DM of a system does not depend on time. Usually the stability of a steady state is achieved by putting the considered system in a contact with a reservoir - another system which is big enough so that its characteristics can not be modified by the considered system. The reservoir adds a compensating term into the von Neumann equation for the DM of the system:[2]

$$\frac{\mathrm{d}\hat{\varrho}}{\mathrm{d}t} = \left.\frac{\partial\hat{\varrho}}{\partial t}\right|_{\text{int}} + \left.\frac{\partial\hat{\varrho}}{\partial t}\right|_{\text{ext}} = \frac{i}{\hbar}\left[\hat{\varrho},\hat{H}\right] + \left.\frac{\partial\hat{\varrho}}{\partial t}\right|_{\text{ext}} = 0. \tag{10.1}$$

In (10.1), $\left.\frac{\partial\hat{\varrho}}{\partial t}\right|_{\text{ext}}$ characterizes the power of the connection of the system to the reservoir, necessary to maintain non-equilibrium steady state with a given energy distribution. This power can be estimated as $(\tilde{T}-T)/t_{\text{r}}$, where T and $\tilde{T}$ are the minimal and the maximal scale of the energy distribution function f_E, and t_{r} is the time the system needs to reach equilibrium due to the relaxation, if left alone.[3] As it is known [38], $t_{\text{r}} \propto \Lambda^{-2}$, Λ being the dimensionless parameter characterizing the smallness of the interaction in the system. In our case $\Lambda \sim U(p_F)/\nu_{\text{2D}}^{(0)} = \ldots$

The average value of an arbitrary physical quantity $\hat{O}$ can be written in the form

$$O = \mathrm{Sp}\left[\hat{\varrho}\hat{O}\right] = \mathrm{Sp}\left[\hat{\varrho}'\hat{O}'\right] + \mathrm{Sp}\left[\hat{\varrho}''\hat{O}''\right] = O' + O'', \tag{10.2}$$

where $\hat{\varrho}' = \mathcal{P}\hat{\varrho}$ is the diagonal part of the DM $\hat{\varrho}$, and $\hat{\varrho}'' = (1-\mathcal{P})\hat{\varrho}$ is the off-diagonal part; $\mathcal{P}$ is the projector extracting the diagonal part of an operator. Analogously, $\hat{O}' = \mathcal{P}\hat{O}$ and $\hat{O}'' = (1-\mathcal{P})\hat{O}$.

$\hat{\varrho}'$ represents quasiequilibrium part [38] of the complete DM $\hat{\varrho}$. It has the maximal entropy possible for the given energy distribution f_E. With $\hat{\varrho}'$ one can formally calculate thermodynamic functions like grand thermodynamic potential Ω and use usual thermodynamic formulas for the calculation of physical quantities. This is the reason why one can call the term $O' = \mathrm{Sp}\left[\hat{\varrho}'\hat{O}'\right]$ in (10.2) quasiequilibrium one. E.g., for quasiequilibrium part of the current we would have

$$\vec{j}' = \mathrm{Sp}\left[\hat{\varrho}'\hat{\vec{j}}\right] = -\frac{\partial\Omega}{\partial\vec{A}}. \tag{10.3}$$

In equilibrium only diagonal matrix elements of a physical quantity enter into the expression for its average value, and quasiequilibrium part of a physical quantity is equal to its real value.

10.2 Thermodynamic current: when interaction is taken into account

About persistent current: cond-mat/0512044, 0706.3369, 0704.1264, 0804.0702.

[1] См. также условие стационарности на стр. [96]47.

[2] For any Hamiltonian, there are infinitely many examples of non-equilibrium but stationary DM: in the SQ, let $\hat{\varrho} = f(\hat{H})$, where f is an arbitrary function. Then $\left[\hat{\varrho},\hat{H}\right] = 0$. (©DL.)

[3] Возражение ВК: Из того, что в рассматриваемой функции распределения энергии есть только 1 масштаб, не значит, что она равновесна. Но возражение это спустя 5 лет представляется мне формальным. Моё же предположение о нескольких масштабах напротив, вполне логично и обосновано. На мой взгляд, ВК просто si stava aggrappando allo specchio, как говорят итальянцы. Возможная причина – аллергия на школу Боголюбова и Бонч-Бруевича у школы Ландау.

The quasi-equilibrium current studied in this section is calculated in [95] using (10.3). STOP Для равновесного тока без взаимодействия есть простое качественное объяснение persistent current в терминах одночастичных энергетических уровней: внешнее поле $B \neq 0$ расщепляет уровни, делая различной энергию электронов с разными L_z, так что остаётся один нескомпенсированный уровень. Неплохо-бы придумать что-нибудь вроде этого для рассмотренного в этой секции случая со взаимодействием.

In this section we use Poisson summation with the same notations as in sec. 8.12. The summation itself is omitted, only Poisson amplitudes (numbered with $\vec{n}$) are calculated and discussed. The methods used here are the same as in sec. 8.12, but explained more briefly. By the word "*conjugation*" I mean the following substitutions: R↔A, $U_K \to -U_K$, while cooperons and diffusons change the frequency sign. For convenience, all the $\left(2\pi\nu_0\tau^2\right)^{-1}$ coming from cooperons and diffusons are included in vertices.

The Hartree and Fock diagrams are given by the following expressions (correspondingly[4]):

$$-i\sum_{kk'} U_{kk'}(0)G(p)\tilde{\gamma}^k G(p)\,\mathrm{Sp}\left[\int \mathrm{d}^d q G(q)\gamma^{k'}\right], \quad i\sum_{kk'}\int \mathrm{d}^d q U_{kk'}(q)G(p)\tilde{\gamma}^k G(p-q)\gamma^{k'}G(p).$$

With the help of *Mathematica*[5] we are able to generate and to select diagrams automatically, see sec. 12.

STOP По всему тексту выражать ток в импульсном пространстве через оператор тока. – Боюсь, из-за этого я в этой секции потерял двойку.

Here I just like in sec. 8.12, I tend to treat cooperons and diffusons in time representation. When holds (9.5) with some arbitrary h_E,

$$\vec{j}(\vec{r}) = \int \mathrm{d}E f_T(E)\vec{j}(\vec{r},E), \quad f_T(E) = \frac{1-h_E}{2}, \tag{10.4}$$

$$\vec{j}(\vec{r},E) = \frac{e\hbar}{2m}\lim_{\vec{r}'\to\vec{r}}(\vec{\nabla}_{\vec{r}} - \vec{\nabla}_{\vec{r}'} - 2ie\vec{A})\left[G_R - G_A\right](\vec{r},\vec{r}';E). \tag{10.5}$$

When G depends only on the difference of its coordinates, (10.5) simplifies to[6]

$$\vec{j}(\vec{r},E) = \vec{j}(E) = ie\hbar\frac{1}{V}\sum_{\vec{p}_n}\vec{v}\left[G_R(\vec{p}_n,E) - G_A(\vec{p}_n,E)\right] = -\frac{1}{V}\sum_{\vec{p}_n}\hat{\vec{j}}(\vec{p})\left[G_R(\vec{p}_n,E) - G_A(\vec{p}_n,E)\right]. \tag{10.6}$$

Out of the equilibrium, together with (10.4), also the contribution (11.8) has to be considered. In equilibrium $f_T(E)$ is just a Fermi distribution function, see (8.27). We have

$$\int \mathrm{d}E\vec{j}(\vec{r},E) = 0, \tag{10.7}$$

due to the fact that $\int \mathrm{d}E\delta G_{R/A}(E) = 0$ which in its turn follows from the statement that $\delta G_{R/A}$ have retarded/advanced analytical structure and that corresponding integrals do converge (unlike those for $G_{R/A}$).

For $G_R - G_A$ we have 4 initial diagrams (+4 *conjugated* ones). Two of them contain only G_R or only G_A so that one can not insert cooperon or diffuson lines in them. Due to this fact they can not depend on $\vec{A}$ and thus cannot give any contribution to the current. Then we have 1 Hartree and 1 Fock diagram left.

One can note that major part of the diagrams (after adding cooperon and diffuson lines) is given by the Fock diagrams.

We have 14 diagrams with the number of cooperon and diffuson loops ≤ 1. According to the estimates (see `estimates.nb`), main contribution is given by diagrams of Ambegaokar & Eckern (2 Hartree and 2 Fock type ones) and the one we suspect to be next most important, see fig. 10.1.

In this section we use (8.27) for calculations.

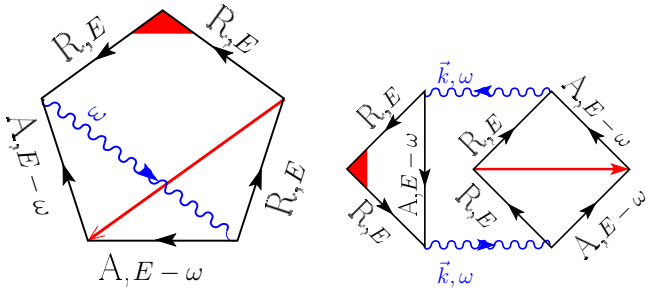
10.3 The renormalization of the potential in the Cooper channel

See also [97]. What we consider here is actually the Cooper channel, see sec. **??**.

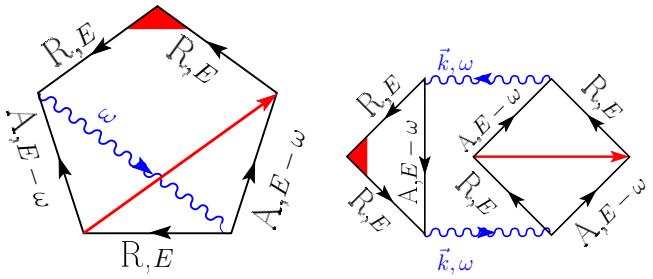
[4] Note that Sp is taken also on spin degree of freedom.

[5] See files `getAvrgdDiags.nb`, `tunDensity.nb` and `diagrams.m`

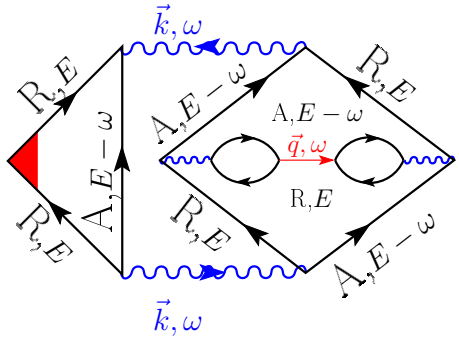
[6] Take a look on general expression for the universal current in [20].



(a) Ambegaokar & Eckern Hartree diagrams with the coefficient $ih_{E-\omega}U_{\mathrm{R}}(0)$.



(b) Ambegaokar & Eckern Fock diagrams with the coeficient $-ih_{E-\omega}U_{\mathrm{R}}(\omega)/2$. One can note that (if the momentum transfered via the interaction line is large) they are equal to the ones from fig. 10.1(a).



(c) Next important diagram with the coeficient $-ih_{E-\omega}U_{\mathrm{R}}(\omega)/2$.

Figure 10.1: The calculated diagrams for the current. The triangulars are $= -4\pi\nu D\tau^3\vec{k}/l$. Every diagram has its complex conjugated "sister" with the opposite sign.

Consider diagrams on the right in fig. 10.1(a) and 10.1(b). Together with them we can (and must!) consider more complicated diagrams obtained by inserting interaction between impurity lines of a cooperon many times.

So, strictly speaking, we thus insert between the cooperons one term from the first order perturbation theory expression for GG. Why we insert **only** this particular term? VK: because other terms manifest themselves in the appearance of the dephasing time τ_φ in the denominator of the cooperon. This additional term in 2D is small like T/g, so that the effects of these terms can be ignored here. The considered term is selected by the fact that for large frequencies Ω, due to the presence of h_Ω in the numerator (together with a Cooperon in the denominator) it goes like $1/|\Omega|$, so that we get logarithm which is big because large values of Ω are important in $\int \mathrm{d}\Omega$. Other terms do not have this h_Ω and hence are much smaller. More precisely: there can be 2 types of elementary "bricks" composing the ladder: without switching, $\binom{\mathrm{R},E\to\mathrm{R},E-\omega}{\mathrm{A},E'\to\mathrm{A},E'+\omega} \propto h_E - h_{E'+\omega}$ and with switching, $\binom{\mathrm{R},E\to\mathrm{A},E-\omega}{\mathrm{A},E'\to\mathrm{R},E'+\omega} \propto h_E - h_{E-\omega}$.

$$T_{\mathrm{F}} \sim 3\times 10^4\mathrm{K}, \quad Dp_{\mathrm{F}}^2 \sim 6\times 10^6\mathrm{K}, \quad \hbar/\tau \sim 800\mathrm{K}, \quad T \sim 0.01\mathrm{K}. \tag{10.8}$$

We will have different renormalization for Hartree and Fock cases. See fig. 10.2 and VI, pp. 65-68 for the details.

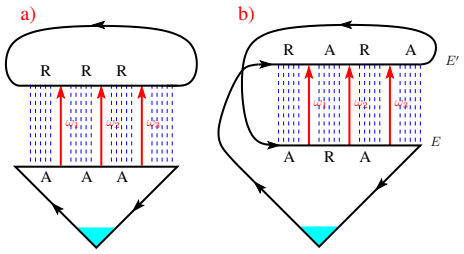


Figure 10.2: Two types of the renormalization in Cooper channel: a) Hartree and b) Fock. Ω_{12} are supposed to be large and $\delta E \equiv E - E'$ – small. On a)$\Omega_3 \equiv \delta E - \Omega_1 - \Omega_2$; on b) $\Omega_3 \equiv -(\Omega_1 + \Omega_2)$.

The key steps for the calculation are:

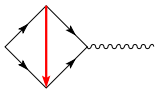
- Every interaction line carries large values of momentum $\sim p_{\mathrm{F}}$. Together with (10.8) this means that we can put $U_{\mathrm{R/A}} = 1/\nu_0$ and $U_{\mathrm{K}} = 0$.
- In both Hartree and Fock cases we have the integration over "number of interaction lines"-1 frequencies. On the right side we have the same $G_{\mathrm{R}}(E)$ and $G_{\mathrm{A}}(E-\omega)$, as on the left side. During the calculation, these our fixed energies are considered to be small in comparison to large Ω: $E, \omega \ll \Omega$.
- an example: a diagram without $G_{\mathrm{R/A}}$ switching (see above) with 3 interaction lines:

$$\frac{1}{i\delta E + Dk^2}\,\frac{h_{E'} - h_{E-\Omega_1}}{i(\delta E - 2\Omega_1) + Dk^2}\,\frac{h_{E'+\Omega_1} - h_{E-(\Omega_1+\Omega_2)}}{i[\delta E - 2(\Omega_1+\Omega_2)] + Dk^2}\,\frac{h_{E'+(\Omega_1+\Omega_2)} - h_E}{i\delta E + Dk^2}$$

We have to perform integration on $(\Omega_1+\Omega_2)$ and Ω_1 leaving only terms that $\sim 1/|\Omega|$ for large Ω.

- an example: a diagram with $G_{R/A}$ switching (see above) with 3 interaction lines:

$$\frac{1}{-i\delta E+Dk^2}\frac{h_E-h_{E-\Omega_1}}{-i(\delta E-2\Omega_1)+Dk^2}\frac{h_{E'+\Omega_1}-h_{E'+(\Omega_1+\Omega_2)}}{-i[\delta E-2(\Omega_1+\Omega_2)]+Dk^2}\frac{h_{E-(\Omega_1+\Omega_2)}-h_{E'}}{-i\delta E+Dk^2}$$

- In the Fock case the calculation of jth self energy  depends on the parity of j, however the result is j - independent.

The result is the same for Hartree and Fock diagrams (see `renormCooperon.nb`): a diagram must be multiplied by $\left[1+\frac{1}{2}\ln\frac{1}{T\tau}\right]^{-1}$ - just as if we had a renormalization of all the components of our (already RPA-renormalized) potential - the suspected reason for the above-mentioned jargon.

The difference from [10] and from what VK has told is that in my case the upper cut-off is $\Omega_{\max}=1/\tau$ (for bigger Ω the vertex and components of the interaction potential begin to depend on it) and not Fermi energy. ВК: всё дело в том, что на больших импульсах мы выходим за пределы диффузионного приближения, поэтому τ появиться не может в любом случае: вместо него имеем Fermi energy. И даже возможно, что (так как это всё - чисто кулоновские дела) – там стоит циклотронная частота вместо фермиевской энергии.

Note also that our calculations are correct in case if

$$! \quad \omega, E, D(\vec{k}+\vec{A})^2 \lesssim (\Omega_i)_{\min} = T.$$

It means that A should not be very big. But we now that we have a periodicity in A so that we can always say that $AL \le \varphi_0$. So if we want our result to hold for all $\vec{A}$, we can just say that the relation $D(\pi/L)^2 \lesssim T$ should hold. For $L = 5\mu$m we get $T \gtrsim 0.03$K.

Note that there is another cause that diminishes the potential constant, see the end of Sec. 9.6.2.

10.4 Calculation of diagrams

From (13.10) and (13.15) follows the recipe of the calculation:

- Use the expressions for Green functions and for the current vertex as if $\vec{A}=0$.
- In general non-equilibrium case [keeping in mind (1.5) and [20]]:

$$\vec{j}(\vec{r},t)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\int\frac{d^dp}{(2\pi)^d}\hat{\vec{j}}(\vec{p})G_K(\vec{p},E). \tag{10.9}$$

In equilibrium case one uses (10.4) with

$$\vec{j}(\vec{r},E)=-\int\frac{d^dp}{(2\pi)^d}\hat{\vec{j}}(\vec{p})\left[G_R(\vec{p},E)-G_A(\vec{p},E)\right]. \tag{10.10}$$

10.4.1 Hartree diagram

The current density is equal to

$$j=\int_{-\infty}^{\infty}\frac{dEd\omega}{(2\pi)^2}f_E\frac{1}{V}\sum_{\vec{k}'_n}(iev)2i\Im ih_{E-\omega}\frac{\Lambda}{\nu}(-4\pi\nu D\tau^3\vec{k}'/l)\frac{(2\pi\nu\tau)^2}{(2\pi\nu\tau^2)^2}\frac{1}{(Dk'^2_n-i\omega)^2}=$$

$$=\int_{-\infty}^{\infty}\frac{dEd\omega}{(2\pi)^2}f_E\frac{1}{V}\sum_{\vec{k}'_n}2evh_{E-\omega}(4\pi\nu D\tau k'/l)\frac{\Lambda}{\nu}\Im i\frac{1}{D^2(k'^2_n+L_\omega^{-2})^2},$$

where $L_\omega = \frac{1+i\,\mathrm{sign}\,\omega}{\sqrt{2}}\sqrt{\frac{D}{|\omega|}}$, and $f_E = (1-h_E)/2$ is the energy distribution function. Now we introduce Poisson summation (for our quasi1D case):

$$\frac{1}{V}\sum_{\vec{k}_n'}\frac{k_n'}{(k_n'^2+L_\omega^{-2})^2} = -\frac{L}{V}\sum_{n\in\mathbb{Z}}\exp\left[2\pi i n\frac{\Phi}{\Phi_0}\right]\int\frac{\mathrm{d}k}{2\pi}\frac{\exp\left[iknL\right]k}{k^2+L_\omega^{-2}} = \frac{2L}{S}\sum_{n>0}\sin\left[2\pi n\frac{\Phi}{\Phi_0}\right]\int\frac{\mathrm{d}k}{2\pi}\frac{\sin\left[knL\right]k}{(k^2+L_\omega^{-2})^2},$$

where

$$\int\frac{\mathrm{d}k}{2\pi}\frac{\sin\left[knL\right]k}{(k^2+L_\omega^{-2})^2} = \frac{L_\omega nL}{4}\exp\left[-\frac{nL}{L_\omega}\right].$$

$$j = \int_{-\infty}^{\infty}\frac{\mathrm{d}E\mathrm{d}\omega}{(2\pi)^2}f_E 8\pi ve\nu h_{E-\omega}\frac{D\tau}{l}\frac{\Lambda}{\nu D^2}\Im i\frac{2}{S}\sum_{n>0}\sin\left[2\pi n\frac{\Phi}{\Phi_0}\right]\frac{L_\omega nL}{4}\exp\left[-\frac{nL}{L_\omega}\right] =$$
$$= -\sum_{n>0}\sin\left[2\pi n\frac{\Phi}{\Phi_0}\right]\frac{4\pi e\Lambda nL}{DS}\Im i\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi}f_E\int_0^{\infty}\frac{\mathrm{d}\omega}{2\pi}\left\{h_{E-\omega}\frac{1+i}{\sqrt{2}}\sqrt{\frac{D}{\omega}}\exp\left[-nL\frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega}{D}}\right]+\right.$$
$$\left.+h_{E+\omega}\frac{1-i}{\sqrt{2}}\sqrt{\frac{D}{\omega}}\exp\left[-nL\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{D}}\right]\right\}.$$

$$\Re\{\ldots\} = \sqrt{\frac{D}{2\omega}}\exp\left[-\frac{nL}{\sqrt{2}}\sqrt{\frac{\omega}{D}}\right](h_{E-\omega}+h_{E+\omega})\left[\cos\left(\frac{nL}{\sqrt{2}}\sqrt{\frac{\omega}{D}}\right)-\sin\left(\frac{nL}{\sqrt{2}}\sqrt{\frac{\omega}{D}}\right)\right] =$$
$$= (h_{E-\omega}+h_{E+\omega})\sqrt{\frac{D}{2\omega}}(\Re+\Im)\exp\left[-nL\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{D}}\right].$$

We have problems with $\int\mathrm{d}E$. To resolve them, we remember that Keldysh technique often produces diagrams that should be completed by adding terms = 0 due to analytic properties.

In case of diagrams in fig. 10.1, we can change their coefficients from $ih_{E-\omega}U_\mathrm{R}(0)$ and $-ih_{E-\omega}U_\mathrm{R}(\omega)/2$ to $i(h_{E-\omega}-1)U_\mathrm{R}(0)$ and $i(1-h_{E-\omega})U_\mathrm{R}(\omega)/2$. This will secure convergence of $\int\mathrm{d}E$. As a consequence, in our expressions $\int\mathrm{d}E f_E\,(h_{E-\omega}+h_{E+\omega})$ gets substituted with

$$\frac{1}{2}\int\mathrm{d}E\left[(1-h_Eh_{E-\omega})+(1-h_Eh_{E+\omega})+(h_{E-\omega}+h_{E+\omega}-2h_E)\right] = 4\tilde{T}_\omega.$$

$$I = jS = -\sum_{n>0}\sin\left[2\pi n\frac{\Phi}{\Phi_0}\right]\frac{8e\Lambda nL}{D}\int_0^{\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{T}_\omega\sqrt{\frac{D}{2\omega}}(\Re+\Im)\exp\left[-nL\frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{D}}\right], \tag{10.11}$$

For similarity with the paper [95] one can perform variable change $\omega\to z = L\sqrt{\frac{\omega}{2D}}$:

$$I = jS = -\sum_{n>0}\sin\left[2\pi n\frac{\Phi}{\Phi_0}\right]\frac{8e\Lambda n}{\pi}\int_0^{\infty}\mathrm{d}z\tilde{T}_z(\Re+\Im)\exp\left[-nz(1+i)\right], \tag{10.12}$$

When one sees (10.12) - like integral, he/she probably immediately says that it does not decay exponentially with temperature. This would be true if $\tilde{T}_\omega$ would not approach constant = $\tilde{T}$ when $\omega\to 0$. The reason is that $\int_0^{\infty}\mathrm{d}z\,(\Re+\Im)\exp\left[-nz(1+i)\right] = 0$, so that only the vicinity of $z=1$ gives contribution to the integral.

Let T be the smallest scale of function h_E. In equilibrium T is the temperature. Then $\delta = T/E_\mathrm{T}$ is the scale of function $\tilde{T}_z$. When δ is large, we can approximate $\tilde{T}_z$ with its expansion over z/δ hoping to obtain the AE for the integral. These attempts fail because $\forall n,m\in\mathbb{N}\ \int_0^{\infty}\mathrm{d}z z^{4m}(\Re+\Im)\exp\left[-nz(1+i)\right] = 0$. We conclude that the temperature-dependence of the AE-current is non-analytic. As it was numerically shown in [95] for the equilibrium case, it is very similar to exponential with the characteristic scale given by the Thouless energy.

Let us use model distribution function $h_E = \frac{1}{2}\left[\tanh\frac{E+V/2}{2T}+\tanh\frac{E-V/2}{2T}\right]$ to study properties of thermodynamic current out of equilibrium. When $T\to 0$, the effective temperature is given by (9.40), and thermodynamic current remains finite no matter how much we increase $\tilde{T}$. This is the illustration of the fact that it is the smallest scale of the distribution function that governs the decay of the thermodynamic current.

Chapter 11

Non-equilibrium: off-diagonal current terms

Take a look on: arXiv/0704.1704.

В случае рассматриваемого двухступенчатого распределения, точно ли не изменятся выражения для $G_{R/A}$? In this section we study essentially non-equilibrium part of the current $\vec{j}''$. See also [91, 88]. This is the simpliest case of a non-equilibrium steady state, since the (quasiclassical) distribution function $f = (1-h)/2$ is spatially homogeneous. Does this automatically implies that f is isotropic in the momentum space? I think not, otherwise the current would be zero. On the other hand, in our calculation we ignore the momentum-anisotropy of f (see a nice picture on p. [48]347). How can we then be sure that we get the correct result? We know that the final result must homogeneous due to the current conservation. And we get a homogeneous result. Someone might argue that we have missed an additional contribution comming from the momentum-anisotropy of f. But as it is argued in [98] and in sec. **??**, this additional contribution disappears, after we integrate the result over the volume of the sample. The latter we are allowed to do because of the current conservation. So, I think the result of this section is correct, as well as [91, 88].

11.1 Calculating diagrams from singlet and triplet channels

STOP Another example of extraction of singlet and triplet channel out of the general Coulomb interaction: arXiv/0801.2139. Здесь мы использовали формулы для диффузона, полученные с учётом первого члена разложения. The principal differences from the above-considered Ambegaokar-Eckern diagrams are:

- In Ambegaokar-Eckern diagrams, large values of momentum were important; this guaranteed their significance.
- Ambegaokar-Eckern diagrams essentially do not have zero-frequency cooperon. The consequences of this formal difference are discussed later on.

Идея ВК: Величины, вычисленные с учётом зависимости ν_E, можно получить заменой τ, D и т. п. на τ_E, D_E. Таким образом, нам не нужно предположение о малости x.

11.1.1 Singlet channel

Наша исходная диаграмма на рис. 11.1 на первый взгляд кажется диаграммрй **нулевого** порядка по взаимодействию, т. к. взаимодействие там – на малых импульсах, то есть оно – экранированное и от (малой) константы взаимодействия не зависит. Но когда мы смотрим на коэффициент при этой диаграмме, видим, что на самом деле при $U_R = U_A$ и $U_K = 0$ он равен нулю. Эти равенства нарушаются лишь начиная со второго порядка, и потому наряду с этой нашей первоначальной диаграммой мы должны рассмотреть все возможные диаграммы вплоть до второго порядка по взаимодействию включительно.

STOP А что мешает одеть второй конец вершины взаимодействием вместе с диффузоном? –Ничего! Получу ли я тогда результат, больший, чем на рис. 11.1? Комментарий ВК: $\Re U$ не может переключить тип гриновской функции, а $\Im U$ для нас плох, так как он нам испортит мнимую часть. Да и вообще – рассматривать такие диаграммы – значит идти по порочному пути Амбегаокара и Экерна. The singlet channel is represented by the diagram in fig. 11.1 with a coefficient

$$K = \frac{i}{2}\left\{(h_E - h_{E-\omega})\, U_K(\omega) - (1 - h_E h_{E-\omega})\left[U_R(\omega) - U_A(\omega)\right]\right\}. \tag{11.1}$$

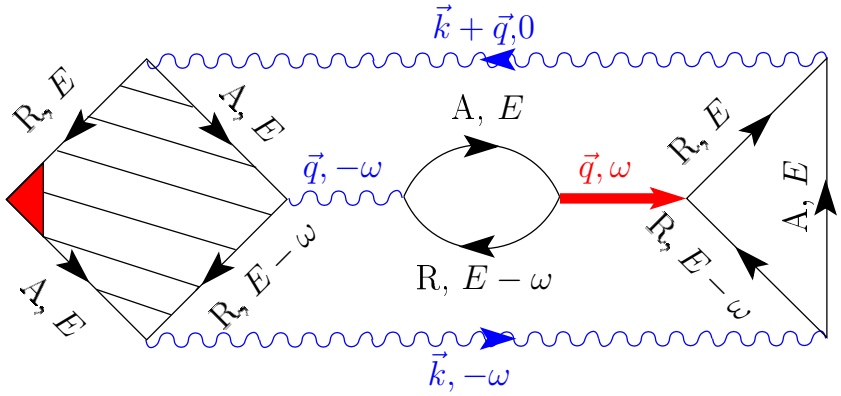


Figure 11.1: A non-equilibrium diagram (singlet channel) for the current with a coefficient given by (11.1). Hikami box= $4\pi i\nu D\tau^4\vec{q}/l$.

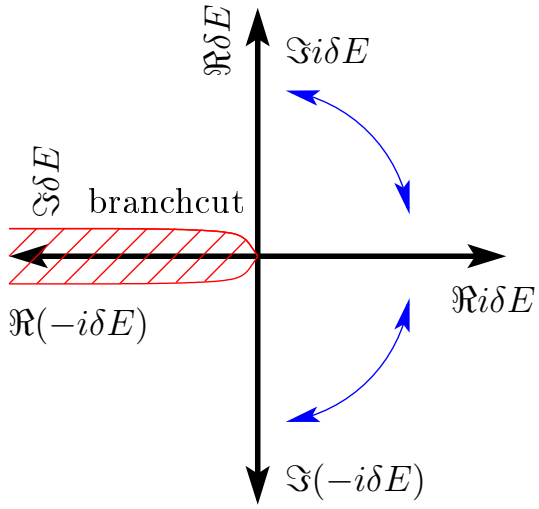


Figure 11.2: The frequency integral in the complex plane.

Note that K changes sign when subjected to the transformation: $E\to E+\omega$ and then $\omega\to-\omega$; this is also confirmed by (11.4) and (11.5). Our general statement is that only off-diagonal elements of the current operator $\hat{\vec{j}}$ give contribution to the answer.[1] This means that if we substitute $\hat{\vec{j}}$ with any diagonal operator, we should get zero result. We are able to prove this in case of $\hat{\vec{j}}\to\hat{1}$. After this substitution, let us use Lehmann representation for $G_{R/A}$ on the ends of our diagram (before the averaging). One can see that $\int dy G_R(x,y)G_A(y,z)\propto G_R(x,z)-G_A(x,z)$. Together with the Green function from the Fock self energy (remember that before the averaging we have two diagrams with that Green function is G_R or G_A) it constitutes the expression, which is invariant to the exchange $E\leftrightarrow E-\omega$. However, the coefficient K (see above) changes sign, so that the result should be zero.

If (9.5) holded for the components of the potential, the coefficient of the diagram on fig. 11.1 would reduce to $K=\frac{i}{2}g(E,\omega)\left[U_R(\omega)-U_A(\omega)\right]$ with

$$g(E,\omega)=2\tilde{T}(\omega)\frac{h_E-h_{E-\omega}}{\omega}-(1-h_Eh_{E-\omega})\underset{\omega\ll\tilde{T}}{\longrightarrow}4\left[\tilde{T}\frac{h'_E}{2}-\frac{1-h_E^2}{4}\right],$$
$$\forall\omega\quad g(E,-\omega)=g(E+\omega,\omega),\qquad\int_{-\infty}^{\infty}\frac{dE}{2\pi}g(E,\omega)=0, \tag{11.2}$$

with $\tilde{T}(\omega)$ defined in (9.38). However, it results that this is not the case [i.e. (9.5) does not hold]. To evaluate (11.1) we use the relations

$$U_{R/A}=\frac{\pi_{A/R}}{\pi_R\pi_A},\quad U_K=-\frac{\pi_K}{\pi_R\pi_A}. \tag{11.3}$$

An important issue is that we can substitute (11.3) to (11.1) with the denominator evaluated without taking into account ν_E - dependence, and thus calculated using simple expressions (9.41). This is due to the fact (see the calculations below) that $\pi_R\pi_A$ (see (9.35)) are E and E' - independent, so that we can rearrange our energy integrals in the manner

$$\int d\omega\frac{1}{\pi_R\pi_A}\int dEdE'R_\omega(E,E')\ldots$$

If one neglects ν_E - dependence under $\int dEdE'$, $R_\omega(E,E')$ will rest the only quantity depending on E and E'; then from (11.6) and (11.2) we see that $\int dEdE'R_\omega(E,E')=0$. So we deduce that the correction to $\pi_R\pi_A$ in the denominator of (11.3), due to the dependence of $\nu_E\neq$ const, lies out of the considered precision. With this argument, using (9.35) and (9.41),

[1] In this sence, the non-equilibrium first-moment of the current is similar to the equilibrium second-moment, where also off-diagonal elements of the current operator participate, see sec. **??**. Note 17.01.2008: See sec. 8.1: I can not give a reliable definition for the off-diagonal MEs of the current operator in coordinate representation (FQ); the situation in the particle-number representation is better, see (8.1),8.2.

we get:

$$K = \frac{1}{2\nu_0 D_0 q^2}\int_{-\infty}^{\infty} dE' R_\omega(E,E')\frac{(D_0q^2)^2+\omega^2}{(D_{E'}q^2)^2+\omega^2}, \tag{11.4}$$

$$R_\omega(E,E') = (h_E - h_{E-\omega})(1-h_{E'}h_{E'-\omega}) - (h_{E'}-h_{E'-\omega})(1-h_Eh_{E-\omega}), \tag{11.5}$$

$$R_\omega(E,E') = -R_\omega(E',E), \quad R_\omega(E) \stackrel{\text{df}}{=} \int dE' R_\omega(E,E') = \frac{g(E,\omega)}{2\omega}. \tag{11.6}$$

The function $R_\omega(E,E')$ from (11.5) is the same as in the kinetic equation [21], where it plays a role of a driving force guiding system to equilibrium - the same what is done in Liuville equation by off-diagonal elements of the density matrix. This is another way to conclude that the contribution we study is given exclusively by off-diagonal elements of both density matrix and current operator. Note that the symmetry group of $R_\omega(E,E')$ is given by 2 generators: (a) $R_\omega(E,E')$ is invariant under transformation $\omega \to -\omega$, $E \to E'-\omega$, $E' \to E-\omega$; (b) it changes sign with $E' \leftrightarrow E$. For singlet channel any of them leads to zero result in case of constant ν; for other diagrams (see fig. 11.3 and 11.4) only (a) proves this statement. The second symmetry (b) leads to $\int dE dE' R_\omega(E,E') = 0$, providing energy conservation (that is, $\int dE \frac{\partial}{\partial t} h_E = 0$) in the kinetic equation [21].

Using (1.16) and (13.36) we obtain the vertex[2]:

$$\left(\frac{-iev}{2}\right)\left(4\pi i\nu_E D_E \tau_E^4 \vec{q}/l_E\right)\frac{1}{(2\pi\nu_E\tau_E^2)^3}(2\pi\nu_0\tau_0)\left[2\pi i\nu_E\tau_E^2\right] = ieD_E\vec{q}, \tag{11.7}$$

STOP Я подозреваю, что в статье [91] я провёл расчёты, подставляя $(-iev)$ вместо $\left(\frac{-iev}{2}\right)$ в (11.7), потеряв таким образом двойку.

Note that for *conjugated* diagram K will change sign, while the vertex (11.7) will be the same, so that my "*conjugation*" in this case has usual (complex-variable) sense. И отсюда сразу следует, что (11.8) есть действительная величина.

Using (9.30), (9.34) and (13.1), we deduce that the contribution to the current from the diagram from fig. 11.1 is equal to

$$S\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{d\omega}{2\pi}\frac{ievl_E}{2\nu_0 dD_0}\times\frac{1}{V}\sum_{\vec{m}\in\mathbb{Z}^d\setminus\{\vec{0}\}}\frac{\vec{q}_{\vec{m}}}{q^2_{\vec{m}}}\int_{-\infty}^{\infty}dE' R_\omega(E,E')\frac{(D_0q^2_{\vec{m}})^2+\omega^2}{(D_{E'}q^2_{\vec{m}})^2+\omega^2}\times$$
$$\frac{1}{D_Eq^2_{\vec{m}}-i\omega}\times\frac{1}{V}\sum_{\vec{n}+\frac{\vec{\Phi}}{-\Phi_0}\in\mathbb{Z}^d}\frac{1}{D_Ek^2_{\vec{n}}-i\omega}\cdot\frac{1}{D_E(\vec{k}_{\vec{n}}+\vec{q}_{\vec{m}})^2} + \text{c.c.} \tag{11.8}$$

We proceed making calculations for quasi one-dimensional case[3], i.e. for the thin ring. Use (13.15), notice that from both exponents $\exp[i\ldots]$ that appear only their imaginary part $i\sin[\ldots]$ survives

$$j = -\sum_{n\geq 1}\sin\left[2\pi n\frac{\Phi}{-\Phi_0}\right]I_n^{(S)}, \quad L_\omega \stackrel{\text{df}}{=} \sqrt{D_E/(-i\omega)}) \Rightarrow L_{-\omega} = L^*_\omega, \tag{11.9}$$

$$I_n^{(S)} = -\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{evl_E}{\nu_0 dD_0}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dE' R_\omega(E,E')\cdot 2\Im\frac{1}{L}\sum_{m\geq 1}\frac{1-e^{-nL/L_\omega}}{\left(D_Eq^2_m-i\omega\right)^3}\cdot\frac{(D_0q^2_m)^2+\omega^2}{(D_{E'}q^2_m)^2+\omega^2}. \tag{11.10}$$

Note that $\sum_n$ diverges.[4] VK: this is due to the failure of the loop expansion in case of small Φ (small Φ correspond to large n). In fact, the expansion parameter is[5] $\frac{\sigma_0}{i\omega+DA^2}$. If $\omega = 0$ and A is small, we thus have problems. In case of $A \to 0$ one should use non-perturbative methods of calculation.

[2] Note that in (11.7) one can substitute $\nu_0\tau_0 \to \nu_E\tau_E$, see (1.22).

[3] Note that the physical dimension is 2 or 3, so one should use 2D or 3D values for physical quantities, like ν_E and D_E. Thus the part "quasi" of the word "quasionedimensionality" means that I perform integrations over "small" momenta (that is, momenta of CD lines and of some interaction lines) as 1D, while I still have to integrate over other momenta (basically, over the "main" momentum of bubbles triangles and Hikami boxes, mostly denoted as p) in 2D or 3D. Another subtle issue here is that in principle for a ring made on a 2D nanostructure (that is, for a strip), BC in the transverse direction are not periodic. This leads to the absence of mode with $q_y = 0$. This is a subtle question, which is discussed in: D. L. Maslov, D. Loss and D. P. Vincenzo, "Conductance fluctuations in the metallic phase of the quantum Hall effect", proceedings of the 22nd int. conference on The Physics of Semiconductors, vol. 2, Vancouver, Canada, August 15-19, 1994. I can't think about this now, I will use quasi-one dimensional model with $q_y = 0$, since cool mesoscopic guys like VK do it. I will think about it later.

[4] Чего-то по происшествии 4 лет из-за этой расходимости появились у меня большие сомнения в правильности (11.10) и объяснения ВК.

[5] Уточнить размерность и как это соотносится с (3.37).

Consider the contribution of small $\omega \lesssim E_T \ll T$ to $\int d\omega$.

$$R_\omega(E,E') \approx \omega \left(\frac{\partial R_\omega(E,E')}{\partial \omega}\bigg|_{\omega=0} \right). \tag{11.11}$$

As one can see from (11.12), the expansion of R in (11.11) in ω/T implies the expansion of the final result in E_T/T.

$$\Im \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega \frac{1-e^{-nL/L_\omega}}{\left(D_E q_m^2 - i\omega\right)^3} \cdot \frac{D_0^2 q^4 + \omega^2}{D_{E'}^2 q^4 + \omega^2} = \frac{D_0^2 - D_{E'}^2}{(D_E + D_{E'})^3} \times \frac{1-e^{-nL|q|\sqrt{\frac{D_{E'}}{D_E}}}}{2q^2}. \tag{11.12}$$

In (11.12) we get zero in case of $D_0 = D_{E'}$; this means that in (11.10) we can neglect E-dependence of all the coefficients to the left from $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}$, because it will result in higher order corrections in $\delta\nu_E/\nu_E$; the same is true for the contribution of $\omega \gtrsim E_T$: it leads to corrections[6] of the order of $\sqrt{E_T/T}$.

Let us define

$$C_n = \frac{6}{\pi^2} \sum_{m \geq 1} \frac{1 - \exp[-2\pi m n]}{m^2}, \quad n > 0, \quad C_\infty = 1. \tag{11.13}$$

$$I_n^{(S)} = C_n \frac{e}{48g} \int_{-\infty}^{\infty} \frac{dE dE'}{2\pi} \frac{\partial R_\omega(E,E')}{\partial \omega}\bigg|_{\omega=0} \frac{\delta D_E}{D_0}, \quad g = \nu DS/L. \tag{11.14}$$

Хотел включить сюда уже проделанные расчёты для двухступенчатого температурного распределения, см. q9p41.nb, да нельзя: ведь выражения для CD линии и Hikami box справедливы лишь при условии, что $\omega \sim E_T \ll T$, то есть мы не можем устремлять температуру к нулю.

В вычислениях можно продвинуться дальше (11.14), подставив

$$\frac{\partial R_\omega(E,E')}{\partial \omega}\bigg|_{\omega=0} = \left(1 - h_{E'}^2\right) h'_E - (\ldots E \leftrightarrow E' \ldots).$$

$$\frac{\partial R_\epsilon(\delta E,\omega)}{\partial \delta E}\bigg|_{\delta E=0} = \left(1 - h^2_{\epsilon - \frac{\omega}{2}}\right) h'_{\epsilon+\frac{\omega}{2}} - (\ldots \omega \to -\omega \ldots),$$

и посчитав один из энергетических интегралов:

$$I_n^{(S)} = -C_n \frac{e}{6g} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{\delta D_E}{D_0} \left[\frac{\tilde{T} h'_E}{2} - \frac{1-h_E^2}{4} \right] = \frac{e}{6g\nu_0} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \nu_E \left[\tilde{T} f'_E + f_E(1-f_E) \right], \tag{11.15}$$

where $f_E = (1-h_E)/2$ is the energy distribution function. The term $f_E(1-f_E)$ can be considered as virtual process: a simultaneous birth of a particle and a hole at the same energy level. It is a limit of the composition of generation and recombination processes $= \frac{1}{2}[f_E(1-f_{E-\omega}) + f_{E-\omega}(1-f_E)], \omega \to 0$

Ещё есть соблазн переписать вычисления в этой секции в тех же энергетических переменных, что и в следующей. Не уверен, что это можно сделать; пока привожу лишь аналогии между переменными в этих двух секциях: $\omega \leftrightarrow \delta E$, $E' \leftrightarrow \epsilon - \omega/2$, $E \leftrightarrow \epsilon + \omega/2$.

11.1.2 Вопросы от Лосса

- Что будет, если мы заменим полярность напряжения на полоске? Изменит ли ток знак? Лосс: “Да, вроде как изменит.”
- Заходит ли ток из полоски в кольцо? Он - равновесный или неравновесный?
- Моя электронно-дырочная интерпретация ПОНРАВИЛАСЬ.

[6] См. первичную версию нашей статьи и [91].

11.1.3 Singlet channel in case of a short-range interaction

The short range interaction has no effect if one does not take spin into account. One could see it in section 10.2, and we must be sure that this happens also out of equilibrium. The expression for the triplet channel would be the same, as in the case of Coulomb interaction, see sec. 11.1.4. The result for the singlet channel will be different:

$$U^{(0)}(q) \stackrel{\text{df}}{=} \frac{\lambda}{\nu_0}, \quad U(q) = \frac{\lambda^2}{\nu_0^2}\pi(q),$$

$$K = \frac{\lambda^2\tau_0}{4\nu_0}\int \mathrm{d}E' R_\omega(E,E')\left[\frac{X}{1-X}+\frac{X^*}{1-X^*}\right] = \frac{\lambda^2}{2\nu_0}\int \mathrm{d}E' R_\omega(E,E')\frac{D_0 q^2}{(D_{E'}q^2)^2+\omega^2}.$$

Note that (in contrast to (11.4), also for $D_{E'} = D_0$ K has a pole at $\omega = iD_{E'}q^2$. Using (11.7), we write the expression for the current:

$$\vec{I} = S\frac{1}{V}\sum_q \int \frac{\mathrm{d}E\mathrm{d}E'\mathrm{d}\omega}{(2\pi)^2} 2\Re ie D_E \vec{q}\cdot K\cdot \frac{1}{D_E q^2 - i\omega}\frac{1}{V}\sum_{k'}\frac{1}{D_E^2(k+q)^2}\frac{1}{k^2+L_\omega^{-2}},$$

$$\frac{1}{V}\sum_{k'} = -\frac{2}{S}\sum_{n>0}\sin\left[2\pi n\frac{\Phi}{-\Phi_0}\right] q\frac{1-\exp\left[-nL/L_\omega\right]}{(D_E q^2 - i\omega)^2}, \tag{11.16}$$

so that

$$I = -\sum_{n>0}\sin\left[2\pi n\frac{\Phi}{-\Phi_0}\right] I_n^{(\mathrm{S})}, \quad I_n^{(\mathrm{S})} = -\int\frac{\mathrm{d}E\mathrm{d}E'}{2\pi}4e\frac{\lambda^2}{\nu_0}\times$$
$$\times\frac{1}{V}\sum_{q>0}\Im\int\frac{\mathrm{d}\omega}{2\pi}R_\omega(E,E')\frac{D_0 D_E q^4}{(D_{E'}q^2)^2+\omega^2}\frac{1-\exp\left[-nL/L_\omega\right]}{(D_E q^2 - i\omega)^3}.$$

$$\Im\int\frac{\mathrm{d}\omega}{2\pi} = \left(\left.\frac{\partial R_\omega(E,E')}{\partial\omega}\right|_{\omega=0}\right)\frac{D_0 D_E}{(D_{E'}+D_E)^3}\frac{1-\exp\left[-2\pi mn\sqrt{\frac{D_{E'}}{D_E}}\right]}{2(2\pi m/L)^2},$$

Expanding $\frac{D}{2(D+D')^3}$ and anti-symmetrizing it by ω, we get $-\frac{\delta D-\delta D'}{16D_0^3}$ instead.

$$I_n^{(\mathrm{S})} = -\frac{e\lambda^2}{12g}\int\frac{\mathrm{d}E\mathrm{d}E'}{2\pi}\left(\left.\frac{\partial R_\omega(E,E')}{\partial\omega}\right|_{\omega=0}\right)\frac{D_0^2 D_E}{(D_E+D_{E'})^3}C_n = -\lambda^2\frac{e}{6g\nu_0}\int\frac{\mathrm{d}E}{2\pi}\nu_E\left[\frac{\tilde{T}h'_E}{2}-\frac{1-h_E^2}{4}\right]. \tag{11.17}$$

11.1.4 Aleiner's diagrams

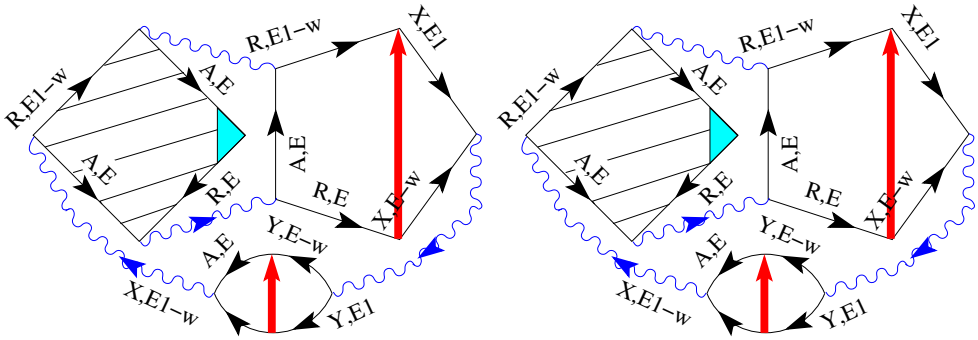
In section 11.1.1 at first glance it seems that all the diagrams beginning from the first order of the interaction were studied (the interaction line in fig. 11.1 means complete RPA series). However, because the coefficient for the diagram in fig. 11.1 approaches zero for the unscreened Coulomb interaction (for which $U_\mathrm{R} = U_\mathrm{A}$ and $U_\mathrm{K} = 0$), really that series of diagrams start from the second order of the interaction. That is why one has to search for important dressed diagrams not only in the first but also in the second order of perturbation. This is the essence of Aleiner's objection, see fig. 11.3.

Without taking $\nu(\xi)$ dependence into account all three (superconducting, triplet and mixed) channels give zero result, and the reasoning follows:

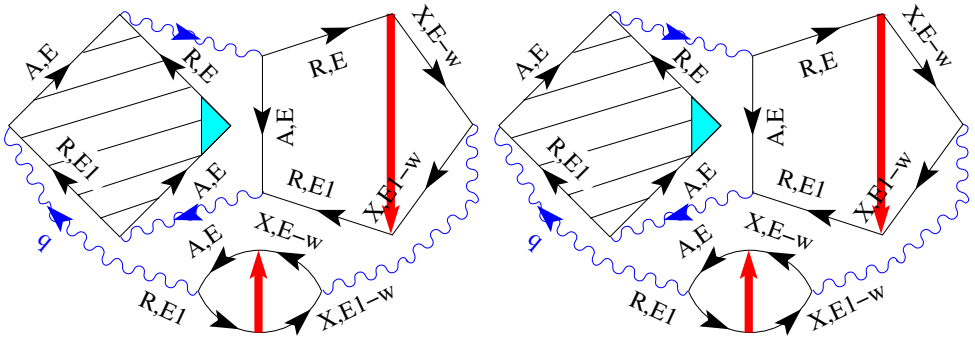
- It is convenient to change variables from $\{E, E', \omega\}$ to $\{\epsilon, \delta E, \omega\}$, where $\delta E = E - E'$ and $\epsilon = \frac{E+E'-\omega}{2}$.
- In these new variables the coefficient $R_\epsilon(\delta E, \omega)$ (defined in (11.5)) is odd both in δE and ω:

$$R_\epsilon(\delta E,\omega) = \left(h_{\epsilon+\frac{\delta E+\omega}{2}} - h_{\epsilon+\frac{\delta E-\omega}{2}}\right)\left(1 - h_{\epsilon-\frac{\delta E+\omega}{2}}h_{\epsilon-\frac{\delta E-\omega}{2}}\right) - (\delta E\to -\delta E), \tag{11.18}$$

  so that also the expression for the diagrams is to be antisymmetrized in δE and ω.
- In addition to that antisymmetrization, we must take $2\Im$ of the diagrams. For constant density of states $\nu(E) = \mathrm{const}$ this is equivalent to
  $[(\ldots) - (\ldots\omega\to-\omega, \delta E\to-\delta E\ldots)]$. Together with antisymmetric properties of (11.18) this leads to conclusion that ALL Aleiner's diagrams = 0 in case of constant density of states. This statement is still true if diffusion coefficients depend only on ϵ.

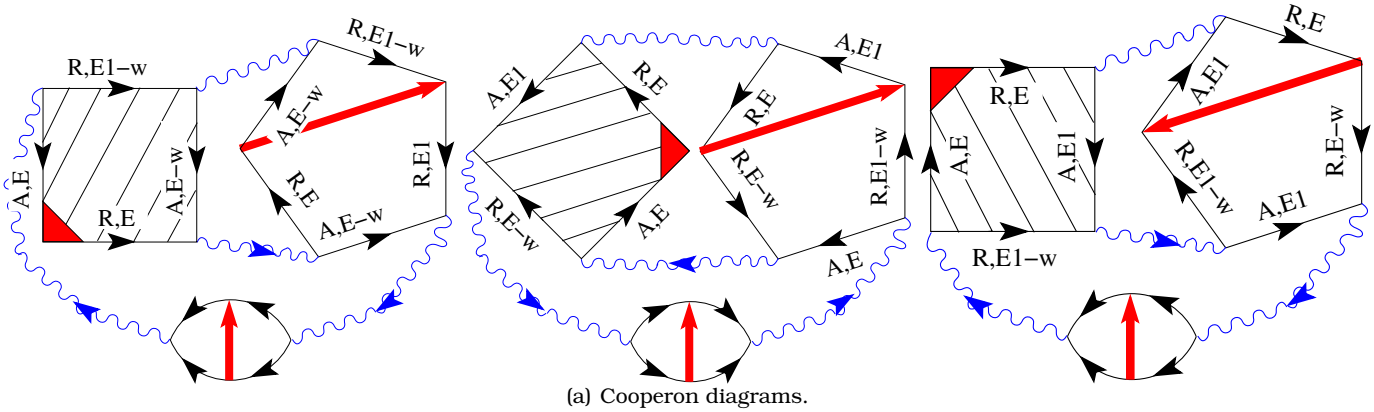


(a) Superconducting channel (a cooperon between 2 interactions). V.K.:"The superconducting channel is a subject for the renormalization in Cooper channel, so that it is diminished by $\ln^2 E_F/T$." Hikami boxes= $-\frac{4\pi i\nu_0 D_0\tau_0^4}{l_0}\left[\vec{k}(1-3xE)-xE\vec{q}\right]$, triangulars= $-2\pi i\nu_0\tau_0(\tau_E-ix/2)$.

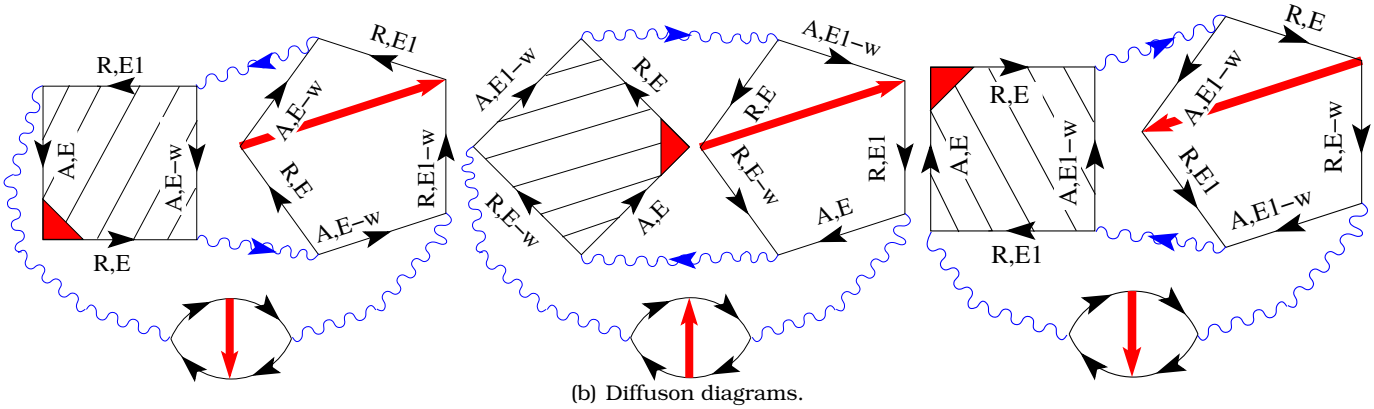


(b) Triplet channel (a diffuson between 2 interactions). Hikami boxes= $4\pi i\nu D\tau^4\vec{q}/l$, triangulars= $-2\pi i\nu\tau^2$.

Figure 11.3: Aleiner's diagrams with large momentum transfer through the interaction. The coefficient is $\frac{\lambda^2}{4\nu_0^2}R_\omega(E,E')$, see (11.5) and (9.36). For each diagram there is another complex conjugate one (if we assume that vertex is real - e.g., $\vec{n}$ (not $\hat{\vec{j}}(\vec{p})$)), which coefficient has the opposite sign. In both channels differences between neighbouring diagrams are highlighted. В выражении для Hikami box появляется $\tilde{\tau}$ из-за пунктирных линий, см. (1.10),1.4. В треугольнике пунктирных линий быть не может, а потому нет и $\tilde{\tau}$.



(a) Cooperon diagrams.



(b) Diffuson diagrams.

Figure 11.4: Aleiner's "mixed" diagrams. The coefficient is $\frac{\lambda^2}{4\nu_0^2} R_\omega(E, E')$, see (11.5) and (9.36).

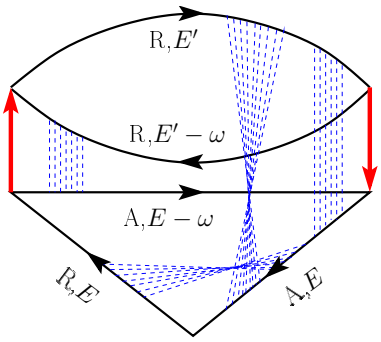


Figure 11.5: The first diagram from Fig. 11.3(b).

In our energy integrals, energy variables can be confined by two energy scales: $E_T \sim 10$mK or $T \gg E_T$. We will call such energy variables correspondingly "small" and "large". As for ϵ, it is surely a large variable, because diffusons and cooperons do not have poles on it. From this we conclude that the integration area where both $\delta E, \omega \ll T$ give zero contribution to the result (because in this case $\delta E, \omega \ll \epsilon \sim T$ so that $D_{\epsilon\pm\delta E} \approx D_{\epsilon\pm\omega} \approx D_\epsilon$).

Note that since at the moment we do not have reliable method for calculating Hikami boxes in the general case when it is not true that $|E - E'| \ll |E + E'|$, we are able to calculate only triplet channel. Other diagrams can only be estimated and fortunately they give minor contributions.

In the superconducting channel the main contribution origins from the zero frequency cooperon, and both frequencies appear to be large. The contribution of the zone with one small frequency is small just because the measure of this zone is small compared with that of the main contribution.

Mixed diagrams from fig. 11.4 are estimated to be $(E_T/T)^3$ and superconducting ones - $(E_T/T)^2$ smaller then those from the triplet channel. In addition to this smallness the superconducting channel fig. 11.3(a) acquires additional one due to the renormalization of the potential (see sec. 10.3).

The triplet channel is special because it has three CD lines with the same energy $\pm\delta E$.

The vertex:

$$\left(-\frac{iev}{2}\right)\frac{(4\pi i\nu_E D_E\tau_E^4\vec{q}/l_E)(-2\pi i\nu_E\tau_E^2)(2\pi\nu_0\tau_0)^3}{(2\pi\nu_E\tau_E^2)^3(2\pi\nu_{E-\omega}\tau_{E-\omega}^2)} = -2\pi iev_{E-\omega}D_E\vec{q}.$$

$$\vec{I} = S\frac{1}{V}\sum_q\int\frac{dEdE'd\omega}{(2\pi)^3}2\Re(-2\pi iev_{E-\omega}D_Eq)\frac{\lambda^2}{\nu_0^2}\times\frac{1}{D_Eq^2+i\delta E}\left[\frac{1}{D_{E-\omega}q^2+i\delta E}+\text{c.c.}\right]\times\frac{1}{V}\sum_{k'}\frac{1}{D_E^2(k+q)^2}\frac{1}{k^2+L_{\delta E}^{-2}},$$

where $L_{\delta E} = \sqrt{D_E/(i\delta E)}$. The sum $\frac{1}{V}\sum_{k'}$ is given by (11.16) with ω changed to $-\delta E$, so that we can write our current in a usual form (11.9) with

$$I_n^{(T)} = \frac{4}{V}\sum_{q>0}\int\frac{d\epsilon d\omega}{2\pi}e\nu_0\frac{\lambda^2}{\nu_0^2}\Im\int\frac{d\delta E}{2\pi}R_\epsilon(\delta E,\omega)\frac{D_0D_Eq^4}{(D_{E-\omega}q^2)^2+\delta E^2}\frac{1-\exp\left[-nL/L_{\delta E}\right]}{(D_Eq^2+i\delta E)^3}. \tag{11.19}$$

We close the integration path, avoiding crossing the branchcut (see fig. 11.2), and the result is given, as usually by the poles in the region $-\frac{\pi}{2} < \arg i\delta E < \frac{\pi}{2}$. Since $\exp[-2\pi] < 0.002$ we neglect the exponential term in (11.19):

$$\Im\int\frac{d\delta E}{2\pi} = \frac{D_0D_E}{2(D_E+D_{E-\omega})^3}\cdot\frac{L^2}{(2\pi)^2}\cdot\frac{\pi^2}{6}\left(\frac{\partial R_\epsilon(\delta E,\omega)}{\partial\delta E}\bigg|_{\delta E=0}\right),$$

$$I_n^{(T)} = \frac{e\lambda^2D_0^2}{6g}\int\frac{d\epsilon d\omega}{2\pi}\frac{D_E}{2(D_E+D_{E-\omega})^3}\left(\frac{\partial R_\epsilon(\delta E,\omega)}{\partial\delta E}\bigg|_{\delta E=0}\right).$$

Then let us substitute $E = \epsilon + \frac{\omega}{2} + \frac{\delta E}{2} \approx \epsilon + \frac{\omega}{2}$ and $E - \omega \approx \epsilon - \frac{\omega}{2}$. This approximation is valid because the region $\{\delta E, \omega | \delta E, \omega \lesssim E_T \ll T\}$ is negligibly small in our whole integration space so that we can drop δE from the expressions for E and $E - \omega$. Then we perform variable change: $\epsilon + \frac{\omega}{2} \to E$, $\epsilon - \frac{\omega}{2} \to E'$. Expanding $\frac{D}{2(D+D')^3}$ and antisymmetrizing it by ω, we get $-\frac{\delta D - \delta D'}{16D_0^3}$ instead, so that

$$I_n^{(T)} = -\frac{e\lambda^2}{48g}\int\frac{dEdE'}{2\pi}\frac{\delta D_E}{D_0}\left(\frac{\partial R_\omega(E,E')}{\partial\omega}\bigg|_{\omega=0}\right) = \frac{e\lambda^2}{6g\nu_0}\int\frac{dE}{2\pi}\nu_E\left[\frac{\tilde{T}h_E'}{2}-\frac{1-h_E^2}{4}\right]. \tag{11.20}$$

The second part of (11.20) has the same structure as singlet channel (11.14); however, it has different sign (see (11.10). In case of short range potential (11.20) is cancelled by (11.17), and this is the manifestation of the fact that short range interaction gives zero effect for fermions if one does not take spin into account.

11.2 Final result for the non-equilibrium current

$$j'' = -\sum_{n\geq 1} \sin\left[2\pi n \frac{\Phi}{-\Phi_0}\right] I_n, \tag{11.21}$$

$$I_n = I_n^{(S)} + I_n^{(T)} = \left(1-\lambda^2\right) \frac{e\tilde{T}}{6g\nu_0} \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} \nu_E \left[\frac{f_E(1-f_E)}{\tilde{T}} + \frac{\partial f_E}{\partial E}\right], \tag{11.22}$$

where $f_E = (1-h_E)/2$ is the energy distribution function.

The result (11.22) is valid for spinless electrons. In case when electrons have spin, λ^2 must be substituted with $3\lambda^2$; see the presentation of my thesis [88]. The reason is that when we apply (13.37) to the ladder-like diagram of the triplet channel, we get 4 terms. From these four three give each equal contribution $\propto \lambda^2$, and the last contributes to the singlet channel. However, in the singlet channel it is small compared to the main contribution calculated in sec. 11.1.1, where Coulomb interaction transfers small momentum. So taking into account spin does not change the expression for the singlet channel, but triplicates the expression for the triplet channel.

Notice that in our case the role of ν_E dependence is not to break the electron-hole symmetry (like in case of thermoelectric/acoustic effects and Coulomb drag): also with an even ν_E dependence (11.22) gives non-zero result. Added 18.08.2005: We thought about two possible sources of ν_E: ZBA (which we did not like) and Kondo effect. However, I think that what we need is actually τ_E dependence, which exists anyway according to p. [48]439. So may be we don't need Kondo?

If someone (nor myself nor VK) just sets interaction to zero in (11.22), he does not get zero. But on the other hand, if he looks to the beginning of the calculation, he notices that all my diagrams on fig. 11.1 and fig. 11.3 contain interaction lines, and thus are inexistent in case when there is no interaction! Thus the current is a non-analytical function of the interaction strength. This point is already very interesting. It questions the results which are obtained for the disordered systems, without taking interaction into account. In particular, see sec. **??**.

Herebeneath follows my interpretation based on my understanding of [38] and hardly criticised by V.K.:

The result (11.22) for j'' can be interpreted as follows: On time scales greater than the *synchronization time* $t_s = r_0/v_F$ (where r_0 is the screening radius of the interaction) the considered non-equilibrium system can be described using quasi-equilibrium momentum distribution function $f_{\vec{p}}$.

Among all the possible excitation processes that govern the evolution of $f_{\vec{p}}$, in the thermodynamic limit the main ones are virtual transitions of particles between the same values of momentum – the so-called *diagonal singularity* property. One can see that these processes are described by the first term in the right part of (11.22): $\int_{-\infty}^{\infty} \mathrm{d}E \nu_E f_E(1-f_E) = \frac{1}{V}\sum_{\vec{p}} f_{\vec{p}}(1-f_{\vec{p}})$. The second term is (for weak ν_E - dependence) almost independent of f_E; it compensates the first one in equilibrium. While in equilibrium the two terms in the square brackets of (11.22) cancel each other for any fixed energy E, in a steady state with a constant density of states we would have this cancellation after integrating over energy. In other words, in case of constant density of states $j'' = 0$, so that we have to take into account the effect of its energy dependence.

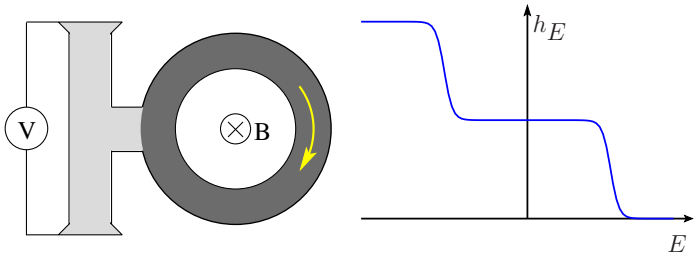
Возражения ВК:

1. Нельзя говорить, что в нашей системе сначала релаксирует импульс, а затем – энергия, ибо система у нам – неоднородная, так что в ней нарушены все возможные симметрии, за исключением обращения времени – симметрии, которая в нашем случае тоже нарушена приложением магнитного поля.

2. Реально диагональной сингулярности в квантовом случае нет (ошибка в [38]): при переходе к термодинамическому пределу остаются не только диагональные элементы $\hat{H}^2$, но и над- и поддиагональные элементы (т.к. такие, у которых одна из частиц переходит в другое состояние – я уже неточно это помню).

11.3 Possible realization

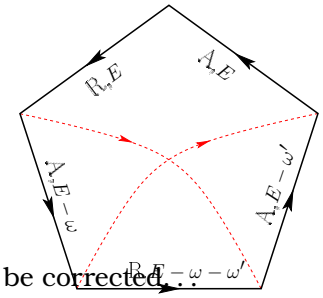
Про яму в плотности состояний из-за взаимодействия – см. cond-mat/0310448, 9810191.
Как показала Швейцария, народ может начать цепляться за эту картинку. Надо это дело расширить и углубить.



Consider a metallic strip with negligibly small interaction between the electrons placed between two electrodes under some voltage, as it is shown in fig. a). This system is in a steady non-equilibrium state with a diagonal density matrix, and it's energy distribution is shown in fig. b) [99]. Attached to a mesoscopic ring, it can be considered as a "non-equilibrium reservoir" that exports its diagonal density matrix into the ring, where the interaction produces off-diagonal MEs. We assume that there is no electric field in the ring; this corresponds to $U \equiv 0$ in the kinetic equation ([21]2.80), from where we conclude that the distribution function is momentum-isotropic. If there were no energy relaxation processes, the momentum symmetric part of the DF inside the ring has to be the same, as in the middle of the strip. This uniquely defines DF, as the momentum-isotropic part of (3.47). In reality we have interaction in the ring, but we assume that it will not change our energy distribution drastically. (though, e.g., interaction will make DF inhomogeneous)

In reality we calculate the current that would arise in a hypothetical situation, when at some moment we switch on interaction in the ring. Whether it is switched also in the rest of the system or not – not so important: the interaction in the strip would slightly reshape its non-equilibrium energy distribution function, leaving it inhomogeneous as before. Once the interaction being switched on in the ring, homogeneous energy distribution in it is no more stable: everywhere in the strip the effective temperature $\tilde{T}$ decreases, and further we are from the contact (and, the weaker the contact with the strip is), the stronger is its decrease. The contact with the strip should not be too strong in order not to increase τ_φ in the ring. This sort of a relaxation finishes with a non-homogeneous non-equilibrium steady state that we (for simplicity) describe as homogeneous.

11.4 To be finished



– to be corrected...

Figure 11.6: One of diagrams from fig. **??**i.

Я уверен, что при постоянной плотности состояний зануляются все диаграммы, кроме рис. **??**i. После перехода к новым энергетическим переменным

$$2\epsilon = 2E - \omega - \omega', \quad E = \epsilon + \omega/2 + \omega'/2, \quad E - \omega = \epsilon - \omega/2 + \omega'/2, \quad E - \omega' = \epsilon + \omega/2 - \omega'/2$$

коэффициент диаграммы на рис. 11.6 запишется в виде

$$-\frac{1}{4}U_{\mathrm{A}}^{\omega'}\left\{\left(h_{\epsilon+\frac{\omega}{2}-\frac{\omega'}{2}}-h_{\epsilon-\frac{\omega}{2}-\frac{\omega'}{2}}\right)\left[\left(U_{\mathrm{R}}^{\omega}-U_{\mathrm{A}}^{\omega}\right)\left(1-h_{\epsilon+\frac{\omega}{2}+\frac{\omega'}{2}}h_{\epsilon-\frac{\omega}{2}+\frac{\omega'}{2}}\right)U_{\mathrm{K}}^{\omega}\left(h_{\epsilon+\frac{\omega}{2}+\frac{\omega'}{2}}-h_{\epsilon-\frac{\omega}{2}+\frac{\omega'}{2}}\right)\right]-\right.$$
$$\left.U_{\mathrm{R}}^{\omega}\left[\left(h_{\epsilon+\frac{\omega}{2}-\frac{\omega'}{2}}-h_{\epsilon-\frac{\omega}{2}-\frac{\omega'}{2}}\right)\left(1-h_{\epsilon+\frac{\omega}{2}+\frac{\omega'}{2}}h_{\epsilon-\frac{\omega}{2}+\frac{\omega'}{2}}\right)-\left(h_{\epsilon+\frac{\omega}{2}-\frac{\omega'}{2}}-h_{\epsilon-\frac{\omega}{2}-\frac{\omega'}{2}}\right)\left(1-h_{\epsilon+\frac{\omega}{2}+\frac{\omega'}{2}}h_{\epsilon-\frac{\omega}{2}+\frac{\omega'}{2}}\right)\right]\right\}$$

11.5 Estimates

We have chosen Kondo effect[7] to be the source of τ_E dependence. There are other sources, see sec. 11.2. $D \sim v_F l$. В длину свободного пробега дают вклад 2 механизма рассеяния: обычный $1/l_0 = n\sigma_0$ и кондовский $\delta(1/l) = n_K\lambda_F^2$:

$$\frac{1}{l} = n\sigma_0 + n_K\lambda_F^2, \quad n_K \ll n_0.$$

Всегда $\sigma_0 < \lambda_F \overset{\text{df}}{=}$"unitary limit". Кондовская примесь достигает максимального сечения рассеяния λ_F^2 при $T = T_K$.

$$\frac{\delta D}{D} \sim \frac{\delta l}{l} \sim n_K\lambda_F^2 l = \frac{n_K}{n_0}\frac{l}{\lambda_F},$$

[7] See, e.g., УФН **178**25 (2008).

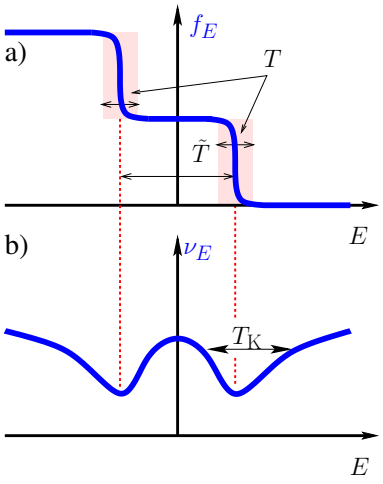


Figure 11.7: Simplified f_E and ν_E dependences. The width of ν_E peaks is $\sim T_K$.

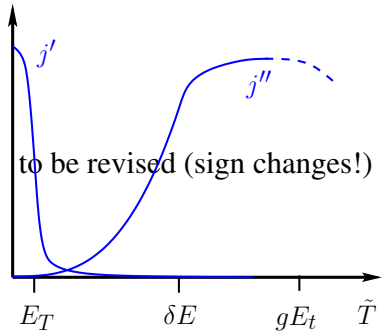


Figure 11.8: Temperature dependence of j' and j''.

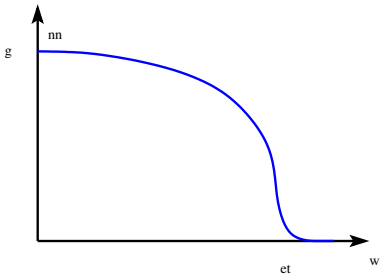


Figure 11.9: Известный результат для мезоскопических флуктуаций плотности состояний.

где $n_0 = \lambda_F^{-3}$ – концентрация электронов $\approx$ концентрация узлов решётки. $1V \sim 10^4 K$, частота: $1K \sim 10^{11}\mathrm{s}^{-1}$, типичная величина persistent current$\sim$ 0.1nA; типичное значение $V \sim 10K$, $L \sim 10\mu m$, $E_T \sim 10mK$, $l \sim 100\lambda_F$

$$I \sim \frac{e}{g}\left(\frac{eV}{h}\right)\frac{\delta D}{D}.$$

Получается, что для того, чтобы иметь $I \sim 0.1nA$, надо $n_K/n_0 \sim 10^{-3} = 1000$ppm. Больше, чем 1000ppm, всобачить трудно. Если специально не чистить материал, обычно концентрация примесей порядка 30 – 50ppm.

Равновесный persistent current (Ambegaokar-Eckern) при $T = 0$ по максимуму даёт $\sim$ 0.1nA (без учёта ренормализации в куперовском канале$\sim \ln^{-1}\left(\frac{E_F}{\max(T,E_T)}\right) \sim 12$). Наш неравновесный ток сравнивается с равновесным при концентрации примесей$\sim$ 100ppm.

11.5.1 Оценка для мезоскопического эффекта.

Зависимость ν_E есть и так, эффект Кондо не нужен. Неусреднённая зависимость ν_E сильно осциллирует; ширина пиков – порядка E_T, их высота $\delta\nu/\nu \sim 1/g$. Оцениваем интеграл $\int \mathrm{d}E \sim V$. Из рис. 11.9 следует, что в оценке тока для мезоскопического эффекта вместо $\delta D/D$ будет стоять $\sqrt{VE_T/g^2}$ (это следует из $(\langle\int_V \mathrm{d}E\delta\nu_E/\nu\rangle)^2 = \int \mathrm{d}E\mathrm{d}E'\delta\nu_E\delta\nu'_E/\nu^2$). То есть отношение среднего тока к мезоскопическому есть

$$g\left(\frac{n_K}{n_0}\right)\frac{l}{\lambda_F}\sqrt{\frac{V}{E_T}} \sim 10^5\frac{n_K}{n_0}.$$

То есть если примесей больше, чем 10ppm, то среднее становится больше флуктуаций. Предел очистки $\sim 1-2$ppm.

11.5.2 Dephasing in the ring

See also: arXiv/0612118, PRB**v77**125312 (antilocalization + τ_φ). The dephasing time is given by the equation [28]:

$$\frac{1}{\tau_\varphi} = \frac{\tilde{T}}{S\nu\sqrt{D}} \int_0^{\tilde{T}} \frac{\mathrm{d}\omega}{\omega^{3/2}}. \tag{11.23}$$

The integral in (11.23) diverges when $\omega \to 0$, so that a cut-off has to be introduced. While for the case of a strip the cut-off is $1/\tau_\varphi$, in case of a ring it must be E_T, due to the (unpublished in 2003) argument of A. D. Mirlin. Thus for a ring we have:

$$\frac{1}{\tau_\varphi} = \frac{\tilde{T}}{S\nu\sqrt{DE_\mathrm{T}}} = \frac{\tilde{T}}{g}, \quad g = \frac{\nu DS}{L}.$$

In the diffusion regime $L_\varphi = \sqrt{D\tau_\varphi}$, so that

$$\frac{L}{L_\varphi} = \frac{1}{\sqrt{E_\mathrm{T}\tau_\varphi}} = \sqrt{\frac{\tilde{T}}{gE_\mathrm{T}}}.$$

См. также §[9]11.4; при $T = 100\mathrm{mK}$ $1/\tau_\varphi \sim 10^{-19}\mathrm{s}^{-1}$; $1/\tau_\varphi \sim T^{2/3}$. В неравновесном случае вместо T следует подставлять $\tilde{T} \sim V$. $1/\tau_\varphi \sim 10^9 \left(\frac{V}{100\mathrm{mK}}\right)^{2/3} \sim 10^{10}\mathrm{s}^{-1}$; $L_\varphi \sim \sqrt{D\tau_\varphi}$; $D \sim 100$, $l \sim 100$Å, $v_F \sim 10^8$cm, так что при $T = 10K$ $L_\varphi \sim 1\mu$m, что ещё приемлемо, то есть нет опасности, что если я слишком увеличу напряжение, то из-за малости L_φ всё умрёт. Надо помнить, что все эти оценки справедливы лишь для $V < T_K$. При $T > 1K$ просыпаются фононы и зависимость τ_φ становится другой. $g \sim 400$, $gE_T \sim 6$K – очень много. При $V > T_K$ зависимость ν_E – слишком размыта, так что рабочая зона есть $V < T_K$. Other literature about dephasing: cond-mat/9712160. Experiment: 0803.0568.

11.6 Cooper channel

It has 4 diagrams: a pair from each of diagrams number 9 and 10. Diagrams number 10: triangle $= -2\pi\nu_0\tau_0 i(1 + \frac{5}{2}\frac{\delta\nu_E}{\nu_0} + \frac{1}{2}\frac{\delta\nu_{E'-\omega}}{\nu_0})$.

Part V

Appendices

Chapter 12

A program for generating and manipulating diagrams

STOP For the spinfull case with SOI but without interaction, the program is rewritten in `maxima` and strongly improved; now it is really a powerful calculation tool [20]. During my Ph.D. thesis I wrote a program in *Mathematica* for generating and manipulating diagrams. It is a part of the thesis and can be downloaded from my home page. Several years later (in spring 2007) I rewrote the program in maxima. Now it not only generates diagrams, but calculates them as well.[1] General statements:

1. The number of loops in a diagram equals to the number of CD lines **minus** the number of vertices (like bubbles or Hikami boxes) **plus** the number of disconnected components of a diagram. The latter is usually one unless we consider correlators or another stuff like that. One can say that every CD line contributes to a smallness, while every vertex leads to a largeness.

   The fact that CD line carry small momentum $k \ll 1/l$ is important from two points of view. First, it means, that if we have 2 diagrams with the same number of loops, the largest is the one having more CD lines. Second, if $k \sim 1/l$, then a Hikami box, from which such a CD line comes out, splits into 2 bubbles. Thus it becomes large, but this is compensated by the fact that CD-line, is small $\sim 1/(p_F l)$, when it transferes large momentum $k \sim p_F$. I need to think more about this. For the moment I am not completely sure, that diagrams with 2 CD-loops are always smaller than the ones with only one CD-loop.

2. An addition of an extra cooperon or diffuson can not decrease the number of loops.

3. (see X, 21-24) Suppose we added N cooperon and/or diffuson lines ($\equiv$ *CD* - lines) to some initial diagram in all the possible ways, and we realize that all the (produced in this way) diagrams have "tails" (see fig. 12.1(a)). This means that no matter how many *CD* - lines we will add more and in what way we do it - any resulting diagram will be with "tails" except for the the only special (not always possible; see fig. 12.1(b)) way of inserting *CD* - lines. This way permits obtaining a no - "tail" diagram, which however will have two extra loops. STOP Утверждение проверено для диффузонов, разъяснить случай куперонов, а также смешанный случай.

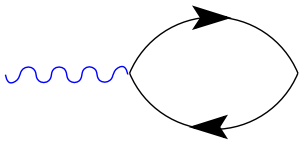
4. Suppose the initial diagram has two disconnected components[2]. Connecting them with a *CD* - line results in an addition of exactly one loop to the entire diagram.

5. Suppose we have a connected diagram with a tail. This tail can be eliminated by a subsequent addition of a *CD* - line; such elimination results in incrementing of the loops number exactly by 1.

Using the program, I perform the following steps (on the example of the SOI-dependent corrections to the charge conductivity from sec. 8):

1. From the original, unaveraged, diagram (bubble), by a subsequent addition of cooperons and diffusons in all possible ways, I produce all different diagrams with the required number of loops.

[1] Some functions from the old program are absent in the new one, in particular, the code for handling cumulants.
[2] The statement is obviously generalized to N disconnected components.



(a) A tail.

(b) Special way of inserting diffusons.

Figure 12.1: The calculated diagrams for the current.

2. We realize that some diagrams may have common parts. [E.g., for convenience I suppose that a cooperon sequence starts with *one* IAL. This first term is the same for both cooperon and diffuson series, and should not be taken into account more than once.] In this way I get an additional list of the diagrams which should be subtracted.from the main contribution.

3. The next step is to connect GFs of the same type with IALs inside HBs in the way, that the principal contribution of all resulting diagrams does not gain additional smallness of the order of $(p_F l)^{-1}$. (Such smallness could occur, e.g., due to the crossing of IALs.)

12.1 Calculation of cumulants

- generalized for the calculation of any quantities which are represented by disconnected diagrams before the end of the averaging procedure.
- the introduction of the group of specific symmetries greatly decreases the number of analyzed expressions. (see `3sigmaSDC1` and `3sigmaDDC2`).
- Inserting firstly the *internal CD* lines (i.e. those in the same disconnected components), I enormously shorten the amount of programs work, because during this first step I can leave only the diagrams with the number of loops ≤maximal number of loops **minus** number of disconnected components+1.

The calculation of the 3rd central conductivity moment in the second loop thus proceed in the following way:

1. The first step: filling the diagram with internal *CD* - lines so that the the number of loops would not be more that $2-2=0$. (because we know that it is impossible to obtain a connected diagram without adding another two loops)
2. The second step. The minimum number of external *CD* - lines necessary to obtain a connected diagram is two. So I add 2 external *CD* - lines in all the possible ways and then drop all the disconnected diagrams. It is not necessary to calculate the number of loops during this step.
3. The last step: adding external *CD* - lines in all the possible ways until it is possible to obtain a diagram with the number of loops ≤ the maximal one (which is 2).

Other ways to reduce the computation time:

- One can make the work of `Union` sufficiently faster by applying it to the diagrams divided in several classes, so that the diagrams from different classes are surely different. Possible characteristics of a class: number of Green functions, *CD* - lines, loops, Hikami boxes.

The time for generating the 2nd loop diagrams for the 3rd conductivity moment without interaction is 4 hours on Pentium 4 1900 MHz.


Добавить сюда доказательства того, что:

- Обувание вершины / взаимодействия увеличивает диаграмму.
- Другие соображения от Лернера, см. #10, стр.. 44.

12.1.1 Additional remarks about interaction.

For n-th order of the perturbation theory, let us draw n+1 Green function's lines. Connect then these lines with n interaction lines - and we get a $n+1$ - particle Green function. Then just produce one - particle Green function from it, connecting $2n$ Green function's ends in all possible ways. This way is convenient if there is a simple algorithm that would give the sign for every such connection.

Chapter 13

Useful relations

In the diffusion approximation we have[1] for dimension d

$$\int \frac{\mathrm{d}\Omega}{\Omega_0} n_i n_j = \frac{\delta_{ij}}{d}.$$

Let us introduce quantity

$$X(q;E,E') \stackrel{\mathrm{df}}{=} \frac{1}{2\pi\nu_0\tilde{\tau}} \frac{1}{V} \sum_{\vec{n}} G_{\mathrm{R}}(\vec{p}_{\vec{n}},E) G_{\mathrm{A}}(\vec{p}_{\vec{n}} - \vec{q}_{\vec{m}},E'), \quad X(q) \equiv X(\vec{q},\omega) = X(q;E,E-\omega) \tag{13.1}$$

From (1.16) we conclude that $X(\vec{0},0) = 1$. From explicit expressions below we see that this is true in 1D, 2D and 3D.

$$X(\vec{q},\omega) = \int \frac{\mathrm{d}\Omega}{\Omega_0} \frac{1}{1 - i(\omega - \vec{v}\vec{q})\tau}, \quad q \ll p_{\mathrm{F}}.$$

$$\text{In 2D} \quad \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \frac{1}{1 + a\cos\varphi} = \frac{1}{\sqrt{1-a^2}}, \quad X(\vec{q},\omega) = \frac{1}{\sqrt{(1-i\omega\tau)^2 + q^2 l^2}}. \tag{13.2}$$

$$\text{In 3D} \quad X(\vec{q},\omega) = \frac{1}{2ilq} \ln \frac{1 - i\omega\tau + ilq}{1 - i\omega\tau - ilq}, \tag{13.3}$$

similar to ([21]4.17).

For $q \gtrsim p_{\mathrm{F}}$, the AE would depend on the particular dispersion law, but there should be a general rule $X(\vec{q},\omega)$ must decrease while q enlarges, and

$$\lim_{q\to\infty} X(\vec{q},\omega) = 0. \tag{13.4}$$

In the diffusion approximation using[2] (1.16), (1.19) and (1.22) we have

$$\begin{gathered} X(\vec{q};E,E') = 1 - \tau_\varepsilon\tau_0/\tilde{\tau} \times \left[D_\varepsilon q^2 - i\omega\right] \approx_{x\to 0} 1 - \tau\left(Dq^2 - i\omega\right), \ \varepsilon = \frac{E+E'}{2}, \\ \nu(\xi) \approx \nu_0(1 + x\xi), \quad \tau_E \equiv \tau(E) \approx \tau_0(1 - xE), \quad l_E = v\tau_E, \quad D_E = \frac{l_E^2}{d\tau_E}. \end{gathered} \tag{13.5}$$

Note that E-dependent corrections of (13.5) make sense if (in case of Cooperon/diffuson) we are near enough to the pole. More precisely, a condition $\omega(\omega + 3D_0q^2)\tau_0 \ll x\varepsilon[2D_0q^2 - i\omega]$ must hold; otherwise we go under the precision of the diffusion approximation. In particular, everything is o.k. when $Dq^2 \sim \omega \ll x\varepsilon/\tau_0$. STOP Замечание ВК: нехорошо здесь сравнивать мнимые величины с действительными.

The result (13.5) is consistent with the general requirement: cooperon's diffusion coefficient is a symmetric function of E and E'; this becomes clear if one considers cooperon plus its complex conjugate (which should certainly be a real quantity).

[1] It can be calculated in the following manner: without loss of generality $\vec{q} = (q,0\ldots0)$, so $(\vec{q}\hat{\vec{n}})^2 = \hat{n}_1^2$. Due to the symmetry $\langle\hat{n}_1^2\rangle = \frac{1}{d}\sum_{i=1}^d \langle\hat{n}_i^2\rangle = \frac{1}{d}\langle\|\hat{\vec{n}}\|^2\rangle = \frac{1}{d}$.

[2] To obtain (13.5), I at first expanded $G_{\mathrm{A}}(\vec{p}-\vec{q},E-\omega)$ in $\vec{q},\omega$; then used (1.16) for the zeroth-order term, then substituted (1.19) and at last integrated by $\mathrm{d}\xi\frac{\mathrm{d}\Omega}{\Omega_0}$.

In 3D

$$\int \frac{\mathrm{d}\Omega}{\Omega_0} n_i^2 n_j^2 = \frac{1+2\delta_{ij}}{15}.$$

$$\int_0^\pi \frac{\mathrm{d}\varphi}{\pi} e^{iz\cos\varphi} = J_0(z), \quad J_0'(z) = -J_1(z). \tag{13.6}$$

A property of the Laplace transformation (DL):

$$\lim_{\omega\to 0} \omega j(\omega) = \lim_{t\to\infty} j(t) \quad \text{if both limits exist.} \tag{13.7}$$

The expression for the current of arbitrary particles[3] and its operator with charge e (see ([60]115.4) and ([60]115.6)):

$$\vec{j}(x) = \frac{ie\hbar}{2m}\left[(\nabla\psi^*(x))\psi(x) - \psi^*(x)(\nabla\psi(x))\right] - \frac{e^2}{mc}\vec{A}(x)\psi^*(x)\psi(x) + \frac{\mu c}{s}\,\mathbf{rot}\left[\psi^*(x)\hat{\vec{s}}\psi(x)\right], \tag{13.8}$$

In a many particle system, WF $\psi(x)$ must be normalized to the number of particles.

$$\vec{j}_{ab}(x) = \frac{ie\hbar}{2m}\left[(\nabla\psi_a^*(x))\psi_b(x) - \psi_a^*(x)(\nabla\psi_b(x))\right] - \frac{e^2}{mc}\vec{A}(x)\psi_a^*(x)\psi_b(x) + \frac{\mu c}{s}\,\mathbf{rot}\left[\psi_a^*(x)\hat{\vec{s}}\psi_b(x)\right]. \tag{13.9}$$

Due to the application of a constant vector potential Green function changes like (see [101])

$$G(\vec{r},\vec{A}) = G(\vec{r},\vec{A}=0)e^{ie\vec{A}\vec{r}}. \tag{13.10}$$

At the Fermi energy ($E=0$) DoS is given by[4]:

$$\text{in 1D } \nu_0 = \frac{m}{2\pi p_\mathrm{F}}, \text{ in 2D } \nu_0 = \frac{m}{2\pi\hbar}, \text{ in 3D } \nu_0 = \frac{mp_\mathrm{F}}{2\pi^2\hbar^2}, \implies \forall d \quad \nu_0 \sim \frac{m}{2\pi\hbar\lambda\!\!\!^{-d-2}}, \quad \lambda\!\!\!^{-} = \frac{\hbar}{p_\mathrm{F}}, \tag{13.11}$$

where in 1D and in 2D the relation "~" may be substituted with "=". Note: the averaged DoS is not necessary equal to the DoS of the free EG. E.g., the averaged DoS in the 2D is energy dependent, contrary to the case of the free EG.[5]

$$G_\mathrm{R}(\vec{r},t) = -\frac{i}{V}\vartheta(t)\sum_p e^{i\vec{p}\vec{r}-i\varepsilon(p)t},$$

$$G_\mathrm{R}(0,t) = -\frac{i}{V}\vartheta(t)\int \frac{\mathrm{d}^d p}{(2\pi)^d} e^{-i\varepsilon(p)t} = \int_{-\infty}^{\infty} \mathrm{d}\Gamma_E e^{-iEt} = \boxed{\text{in 2D}} = -i\vartheta(t)\frac{m}{2\pi}\int_0^\infty \mathrm{d}E e^{-iEt} = \frac{m}{2}\delta(t).$$

$$\delta(\vec{r}) = \frac{1}{(d-1)!}\frac{1}{\Omega_d}\lim_{l\to 0}\frac{e^{-r/l}}{l^d}, \quad \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{13.12}$$

$$\sum_{n\in\mathbb{Z}}\delta(x-n) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x} \quad \Longrightarrow \quad \sum_{n\in\mathbb{Z}} f(n+\xi) = \sum_{m\in\mathbb{Z}} e^{-2\pi i m\xi} \times \int e^{2\pi i m x} f(x)\mathrm{d}x. \tag{13.13}$$

Установить этот факт наверняка! I am sure that (13.13) can be generalized for d dimensions:

$$\sum_{\vec{n}\in\mathbb{Z}^d} f\left(\vec{n}+\vec{\xi}\right) = \sum_{\vec{m}\in\mathbb{Z}^d} e^{-2\pi i\vec{m}\vec{\xi}} \times \int e^{2\pi i\vec{m}\vec{x}} f(\vec{x})\mathrm{d}^d x. \tag{13.14}$$

[3] For the case of electrons, (13.8) is obtained in the classical (with 1st-order corrections in expansion in $1/c$) limit from the Dirac current, see problem [100]15.31. The last term in (13.8) is $\frac{e\hbar}{2mc}\,\mathbf{rot}\,\psi^*\vec{\sigma}\psi$, $\mu = \frac{e\hbar}{2mc}$ (for electrons). At first I thought that in gauge (13.8) is valid only in gauge with scalar potential = 0, but now I think it is general. Since div **rot** ≡ 0, rotor is insignificant in the current conservation relation. I have seen this term coming in the non-relativistic limit of Dirac equation.

[4] The density of states without taking spin into account, see p. 21. Note that these formulas **do not coincide** with two books: p. [46]44 and [102]. Later comment: the reason for this might be that my definition (1.13) might differ by $\hbar$ from the most common one.

[5] This dependence can be ignored in most cases, but sometimes it is important. If this dependence was absent, the identity $\mathrm{Sp}\,\hat{G}_\mathrm{R}^2 = 0$ would *exactly* hold.

Using (13.13), we arrive to a useful relation[6]:

$$\frac{1}{V}\sum_{\vec{n}\in\mathbb{Z}^d} f\left(\vec{q}_{\vec{n}}-\frac{2e}{c}\vec{A}\right)=\sum_{\vec{n}\in\mathbb{Z}^d}\exp\left[2\pi i\vec{n}\frac{\vec{\varphi}}{\varphi_0}\frac{e}{|e|}\right]\times\int\frac{\mathrm{d}^d q}{(2\pi\hbar)^d}\exp\left[\frac{i}{\hbar}\vec{q}\left(\vec{n}\circ\vec{L}\right)\right]f(\vec{q}),$$
$$\vec{L}=\left(L_x,L_y,\ldots\right)^T,\quad \vec{q}_{\vec{n}}=\frac{2\pi\hbar\vec{n}}{\vec{L}},\quad \vec{\varphi}=\frac{\vec{A}\circ\vec{L}}{c},\quad \varphi_0=\frac{\pi\hbar}{|e|}=\frac{h}{2e}. \tag{13.15}$$

For functions of 2 coordinate variables, sometimes it makes sence to work in the mixed momentum-coordinate (Wigner) representation:

$$A(x,y)=\int\frac{\mathrm{d}p}{2\pi}e^{ip(x-y)}\overline{A}\left(\frac{x+y}{2},p\right),\quad \overline{A}(R,p)=\int\mathrm{d}r e^{-ipr}A\left(R+\frac{r}{2},R-\frac{r}{2}\right). \tag{13.16}$$

$$\text{Green's formula}:\quad \int_\Omega(U\delta V-V\delta U)\mathrm{d}\Omega=\int_{\partial\Omega}\left(U\frac{\partial}{\partial\vec{n}}V-V\frac{\partial}{\partial\vec{n}}U\right)\mathrm{d}\vec{S}. \tag{13.17}$$

$$\int_{-\infty}^{\infty}e^{i\vec{p}\vec{z}}(2\vec{p}-\vec{q})_{\alpha\beta}\exp\left[-p^2x^2-y^2(\vec{p}-\vec{q})^2\right]\mathrm{d}^2p=\pi\frac{2(x^2+y^2)\delta_{\alpha\beta}+\left[\vec{q}(x^2-y^2)-i\vec{z}\right]_{\alpha\beta}}{(x^2+y^2)^3}\exp\left[-\frac{x^2y^2q^2+z^2/4}{x^2+y^2}\right]\exp\left[\frac{iy^2\vec{z}\vec{q}}{x^2+y^2}\right]=$$
$$=\frac{2\pi}{\tau_1^2}\exp\left[-\frac{(\tau_1\vec{q}-i\vec{z})^2}{4\tau_1}\right]\exp\left[-\frac{z^2}{2\tau_1}\right]\times\left(\delta_{\alpha\beta}+\frac{1}{2\tau_1}\left[\tau_2\vec{q}-i\vec{z}\right]_{\alpha\beta}\right)\exp\left[\frac{\tau_2^2q^2-2i\tau_2\vec{z}\vec{q}}{4\tau_1}\right],$$

$$\text{where}\qquad x^2+y^2=\tau_1,\qquad x^2-y^2=\tau_2,\qquad x^2=\frac{\tau_1+\tau_2}{2},\qquad y^2=\frac{\tau_1-\tau_2}{2}.$$

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\vec{k}\exp\left[i\left(\vec{L}\circ\vec{m}\right)\vec{k}-Dtk^2\right]=\frac{i\vec{L}\circ\vec{m}}{2Dt}(4\pi Dt)^{-d/2}\exp\left[-\frac{(\vec{L}\circ\vec{m})^2}{4Dt}\right], \tag{13.18}$$
$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\exp\left[i\left(\vec{L}\circ\vec{m}\right)\vec{k}-Dtk^2\right]=(4\pi Dt)^{-d/2}\exp\left[-\frac{(\vec{L}\circ\vec{m})^2}{4Dt}\right]$$

A tetrahedron with a unity edge has height $1/\sqrt{3}$; the area of its side is $\sqrt{3}/4$. The площадь поверхности of an n-dimensional sphere is equal to $S_n=2\pi^{\frac{n+1}{2}}r^n/\Gamma\left(\frac{n+1}{2}\right)$; $S_1=2\pi r$.

Let us use some definitions in Heisenberg (i.e. time) representation from [19] (however, changing signs in them order to have correspondence with [5]):

$$G_{\mathrm{R}}^{(\pm)}(x,x')\equiv-K_{\mathrm{R}}^{(\pm)}(x,x')=-i\vartheta(t-t')\langle[\hat{\psi}(x),\hat{\psi}^\dagger(x')]_\pm\rangle,\quad G_{\mathrm{A}}^{(\pm)}(x,x')\equiv-K_{\mathrm{A}}^{(\pm)}(x,x')=i\vartheta(t'-t)\langle[\hat{\psi}(x),\hat{\psi}^\dagger(x')]_\pm\rangle, \tag{13.19}$$
$$\tilde{G}_{\mathrm{R/A}}(x,x')\equiv-K_{\mathrm{R/A}}(x,x')=\mp i\vartheta\left[\pm(t-t')\right]\langle\hat{\psi}(x)\hat{\psi}^\dagger(x')\rangle,\quad G(x,x')\equiv-K_{\mathrm{C}}(x,x')=-i\langle T[\hat{\psi}(x)\hat{\psi}^\dagger(x')]\rangle.$$

The "bra"-"kets" $\langle\ldots\rangle$ in the above definitions of GFs mean averaging over the unperturbed (though, probably, non-equilibrium) DM:

$$\langle\ldots\rangle\equiv\underset{\mathrm{SQ}}{\mathrm{Sp}}\left[\hat{\rho}^{(0)}\ldots\right], \tag{13.20}$$

where $\mathrm{Sp}_{\mathrm{SQ}}$ is the SQ trace. The latter is explicitly written below in the expression (13.21) for the OP-DM ([19]1.9):[7]

$$\rho_1(\lambda,\lambda')=\sum_{\substack{(n_1,n_2\ldots\geq0)\\(n_1',n_2'\ldots\geq0)}}\langle n_1,n_2\ldots|\hat{\rho}|\ldots n_2',n_1'\rangle\langle n_1',n_2'\ldots|\hat{\psi}^\dagger(\lambda,t)\hat{\psi}(\lambda',t)|\ldots n_2,n_1\rangle=i\eta\lim_{t'\to t}G_{\mathrm{fb}}(\vec{r}',t';\vec{r},t), \tag{13.21}$$

[6] The usual flux quantum is $hc/|e|=4.14\times10^{-7}\mathrm{G}\cdot\mathrm{cm}^2=2\varphi_0$, see ([3]1.1).

[7] Note: ([19]1.9) is nothing but the expression of the OP DM in the FQ representation [see (13.21)] through the elements of the DM в представлении чисел заполнения. It is important (in Keldysh technique) that in the interaction picture (the unperturbed parts of) both $\hat{\rho}$ and $\rho(\lambda,\lambda')$ are time-independent, see the comment after ([19]6.8a).

where $G_{\rm fb}$ is defined in (9.2), and $\langle\vec{n}|\hat{\rho}(t)|\vec{n}\,'\rangle$ are MEs of the DM in the SQ picture [these MEs *decay* when $\max(\sum_i n_i, \sum_i n'_i) \to \infty$, so that (13.21) converges]. From (13.34) we conclude that our OP DM is normalized by the concentration (without spin):[8]

$$\mathrm{Sp}\,\rho_1 \equiv \int \mathrm{d}\lambda \rho_1(\lambda,\lambda) = n. \tag{13.22}$$

From the FQ-OP-DM (13.21) one could try to define a SQ-OP-DM; however I don't know if this is possible in a general (non-equilibrium) case: The resulting SQ-OP-DM should decay when the number of particles infinitely grows; however, the FQ-OP-DM (13.21) does not contain information about such decay.

An important example is when unperturbed system is in equilibrium, so that

$$\hat{\rho}^{(0)} = \exp\left[-(\hat{H}_0 - \mu\hat{N})/T\right] / \mathrm{Sp}\exp\left[-(\hat{H}_0 - \mu\hat{N})/T\right]. \tag{13.23}$$

It is important that DM in (13.19),13.20 is *time-independent*.[9] In the simplest case $\hat{\rho}$ is an equilibrium matrix (13.23); however, it is also possible that $\hat{\rho}$ describes a non-equilibrium steady state – see Sec. 10.1.

Next,

$$G^{(+)}_{\rm R/A} + G^{(-)}_{\rm R/A} = 2\tilde{G}_{\rm R/A}, \quad G_{\rm C} = \tilde{G}_{\rm R} - \eta\tilde{G}_{\rm A}, \tag{13.24}$$

where $\eta = \pm 1$ for the case of bosons and fermions respectively.

$$\text{Connection between [19] and [5] in energy representation:} \quad K_{\rm R/A}(E) = -\frac{1}{2\pi}G_{\rm R/A}(E). \tag{13.25}$$

For fermions, $G_{\rm R/A} \equiv G^{(+)}_{\rm R/A}$ (and for bosons $G_{\rm R/A} \equiv G^{(-)}_{\rm R/A}$) obey the simplest equations, and this must be the reason why just they are usually considered, see ([19]6.2-4). In equilibrium any Green function can be obtained from any other one using spectral function [19].

In a disordered electron system without interaction,

$$\hat{H}_{II} \equiv \hat{T}_{II} + \hat{U}_{II} = \int \mathrm{d}\lambda\mathrm{d}\lambda' \varepsilon(\lambda,\lambda')\hat{\psi}^\dagger(\lambda,t)\hat{\psi}(\lambda',t), \quad \varepsilon^*(\lambda,\lambda') = \varepsilon(\lambda',\lambda)$$

where the time t can be chosen freely, see the note after ([19]6.8a). From here I get an equation of motion for $G_{\rm R}$

$$\begin{gathered}
[\hat{H}_{II}, \hat{\psi}^\dagger(\lambda,t)] = -[\hat{H}_{II}, \hat{\psi}(\lambda,t)]^\dagger = \int \mathrm{d}\lambda' \hat{\psi}^\dagger(\lambda',t)\varepsilon(\lambda',\lambda), \\
i\frac{\partial}{\partial t'}G_{\rm R}(\lambda,t;\lambda',t') + \int \mathrm{d}\lambda'' G_{\rm R}(\lambda,t;\lambda'',t')\varepsilon(\lambda'',\lambda') = -\delta(t-t')\delta(\lambda-\lambda'), \\
(E+i\delta)G^E_{\rm R}(\lambda,\lambda') - \int \mathrm{d}\lambda'' G^E_{\rm R}(\lambda,\lambda'')\varepsilon(\lambda'',\lambda') = \delta(\lambda-\lambda').
\end{gathered} \tag{13.26}$$

Infinitesimal δ may be important because of the singularity of $G^E_{\rm R}$. We get it because the Fourier transformation used in (13.26) is non-standard: $\int_{-\infty}^{\infty} \exp[(iE-\delta)t]$. Eq. (13.26) [and analogous for $G_{\rm A}$] can be rewritten in the operator form (ready for bra-ket notation):

$$\frac{\partial \hat{G}_{R/A}(t,t')}{\partial t'} = i\delta(t-t') + \frac{i}{\hbar}\hat{G}_{\rm R/A}(t,t')\hat{H}.$$

In case when λ is a coordinate, all integrations $\int \mathrm{d}\lambda \equiv \int \mathrm{d}^d r$ must be performed in the volume of the system. Certainly, then all coordinates must be than also restricted within this volume. There is no problem, if this volume is finite, because $\{|\vec{r}\rangle\}_{\vec{r}\in\Omega}$ form a complete ortogonal basis for the Hilbert space of functions, defined within Ω.

If I then define a Green function operator in time domain according to $\langle\lambda|\hat{G}_{\rm R}(t,t')|\lambda'\rangle \stackrel{\rm df}{=} G_{\rm R}(\lambda,t;\lambda',t')$, then I rewrite (13.26) in an operator form as

$$i\frac{\partial}{\partial t'}\hat{G}_{\rm R}(t,t') + \hat{G}_{\rm R}(\hat{T}_I + \hat{U}_I) = -\delta(t-t'), \quad \hat{G}^E_{\rm R}[E - \hat{T}_I - \hat{U}_I] = 1 \tag{13.27}$$

[8] Eq. (13.22) defines the chemical potential μ. For free EG at $T=0$ it can be calculated (so that we know the dependence $\mu(n)$). However, because of the frequency-dependence of τ, even for $T=0$ it is hard (if possible) to calculate (13.22) for the disordered EG (expressing the DM via the averaged GFs). See Sec. 1.5.

[9] If DM $\hat{\rho}$ would be time dependent, a GF would have an additional time argument. Alternatively, one could assign to the DM the time of the operator being averaged. In case of a GF the averaged operator has two times: t and t', so it is unclear which of them would we want to assign to the DM.

where $\hat{T}$ and $\hat{U}$ are operators of kinetic and potential (due to the disorder) energy of an electron. $\hat{G}^E_{\rm R}$ is the Fourier image of $\hat{G}_{\rm R}(t,t')$.

If DM is diagonal in the *same* (SQ) representation, as the Hamiltonian:

$$\langle n_1 \dots n_N|\hat{\rho}(t)|n'_N \dots n'_1\rangle \propto \delta_{n_1,n'_1} \dots \delta_{n_N,n'_N}, \quad \langle n_1 \dots n_N|\hat{H}|n'_N \dots n'_1\rangle \propto \delta_{n_1,n'_1} \dots \delta_{n_N,n'_N}, \tag{13.28}$$

then[10]

$$G^{(+)(0)}_{\rm R/A}(\lambda) = \frac{1}{E-\varepsilon(\lambda)\pm i\delta}, \quad G^{(0)}_{\rm R/A}(\lambda) = \frac{1-n_\lambda}{E-\varepsilon(\lambda)\pm i\delta}, \quad \delta = +0, \tag{13.29}$$

where n_λ is an average occupation number (which is just a Fermi function in equilibrium). For convenience[11] we subtract $\mu = \mu(T)$ from both E and ε in (13.29), see ([19]6.6); then in equilibrium $n_\lambda \equiv n_E = \left(1-\tanh\frac{E}{2T}\right)/2$.

Analogously one obtains for non-interacting bosons in equilibrium:

$$G^{(-)(0)}_{\rm R/A}(\lambda) = \frac{1}{E-\varepsilon(\lambda)\pm i\delta}, \quad G^{(0)}_{\rm R/A}(\lambda) = \frac{1+n_{\rm B}}{E-\varepsilon(\lambda)\pm i\delta}, \quad \delta = +0. \tag{13.30}$$

In these relations $n_{\rm B/F} = \left(e^{\varepsilon/T}\mp 1\right)^{-1}$ are Bose and Fermi distributions. STOP I want to check them for non-equilibrium stationary state! Eqs. (13.29),13.30 are written for the case when both WF and DM are normalized to unity, see p. [19]18-19,45. If I want (13.8) to give me the total current density from all electrons in the system, and (3.22) – the complete conductivity and not just mobility, then I need to normalize WF and DM to the number of particles, see sec. 14.2.

$$\mathrm{res}_{x\to x_0} f(x) = \frac{1}{(n-1)!}\lim_{x\to x_0}\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x-x_0)^n f(x)\right). \tag{13.31}$$

Useful identities for Fermi distribution f_E:

$$\forall E,\omega,T \quad (h_E - h_{E-\omega})\,\mathrm{cth}\,\frac{\omega}{2T} = 1 - h_E h_{E-\omega}, \quad h_E \equiv \tanh\frac{E}{2T} \equiv 1-2f_E, \quad f_E f_{E-\omega} = \frac{f_E+f_{E-\omega}}{2} + \frac{f_E-f_{E-\omega}}{2}\coth\frac{\omega}{2T}. \tag{13.32}$$

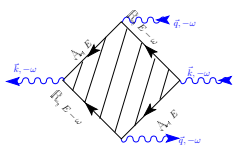
In (13.32) we used (13.11) and (**??**). Moreover, for any h_E and $f_E \equiv (1-h_E)/2$ (i.e. also for out-of-equilibrium systems)

$$\int_{-\infty}^{+\infty}\mathrm{d}E\,(h_E - h_{E-\omega}) = 2\omega, \quad \int_{-\infty}^{+\infty}\mathrm{d}E\,(h_E - h_{E-\omega})\,E = \omega^2 + \mathrm{non-universal\ corrections\ in\ case}\ h_E + h_{-E} \neq 0. \tag{13.33}$$

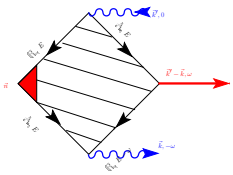
The relation between concentration and density of states is given by (without spin)

$$\forall f_E \quad \int_{-E_{\rm F}}^{\infty} \nu_E f_E \mathrm{d}E = \int_{-E_{\rm F}}^{0} \nu_E \mathrm{d}E = n = \frac{N}{V}, \quad f_E \equiv (1-h_E)/2. \tag{13.34}$$

In equilibrium, (13.34) gives an indirect equation for the function $\mu(T)$; at $T=0$ one obtains $\mu|_{T=0} \equiv E_F = \frac{1}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$. Otherwise (13.34) is just a sum rule, cf. p. [58]322.



$$= 4\pi\nu_0\tau^4\left[i\omega + D\left(k^2+q^2\right)\right]. \tag{13.35}$$



$$= -\frac{4\pi i\nu_0\tau_E^3 D_0\tau_0}{l_0}\left(\vec{k}'-\vec{k}\right) = -4\pi i\nu_0\tau_E^3\left(\vec{k}'-\vec{k}\right)\frac{l_0}{d}. \tag{13.36}$$

[10] Eq. (13.28), IMHO, is one of the restrictions to the validity of the Keldysh technique. Note that this is more than just a requirement that $\left[\hat{\rho}^{(0)},\hat{H}^{(0)}\right] = 0$. DL: it is interesting if this is sufficient for the validity of the Wick theorem.

[11] This makes sense also in a non-equilibrium problem if f_E is not so far from equilibrium.

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\frac{\sin(pnL)}{[L_\omega^{-2}+(p-q)^2][p^2+\delta^2]}=\frac{2e^{-nL\delta}L_\omega^4 q-e^{-nL/L_\omega}L_\omega^3\left\{2L_\omega q\cos(qnL)+\left[1-L_\omega^2(q^2+\delta^2)\right]\sin(qnL)\right\}}{2\left[1+2L_\omega^2(q^2-\delta^2)+L_\omega^4(q^2+\delta^2)^2\right]}.$$

The last equation leads to:

$$\text{v.p.}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\sin knL}{(k+q)^2(k^2+L_\omega^{-2})}=q\frac{1-\exp[-nL/L_\omega]}{(q^2+L_\omega^{-2})^2},\ q=2\pi n/L,\ n\in\mathbb{Z},\ \Re L_\omega>0.$$

$$\text{Pauli matrices:}\quad [\sigma_1,\sigma_2]=2\sigma_1\sigma_2=2i\sigma_3,\quad [\sigma_1,\sigma_3]=-2i\sigma_2,\quad [\sigma_2,\sigma_3]=2i\sigma_1.$$

One[12] useful identity for Pauli matrices [105]:

$$2\delta_{s_1s_2}\delta_{s_3s_4}=\sum_{\alpha=0}^{3}\hat\sigma_\alpha^{s_1s_3}\hat\sigma_\alpha^{s_4s_2}=\sum_{\alpha=0}^{3}\bar\sigma_\alpha^{s_1s_3}\left(\bar\sigma_\alpha^{\dagger}\right)^{s_4s_2},\quad \bar\sigma_\alpha\equiv\sigma_2\sigma_\alpha, \tag{13.37}$$

Note that, unlike σ, matrices $\bar\sigma$ do not form an orthogonal basis.

If I turn the CS[13] counterclockwise by angle φ in (x,y) plane, then coordinates of a particle in the new CS $\vec r\,'$ will be related to it's coordinates in the original CS $\vec r$ as $\vec r\,'=R\vec r$.

In case the rotation occurs in the (x,y) plane, the R is given by

$$R_\varphi^y=(R_{-\varphi}^y)^{-1}=(R_{-\varphi}^y)^T=\begin{pmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{pmatrix},\quad R_\varphi^z=\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{13.38}$$

The connection between rotations in real and spinor spaces is defined in such a way that the magnetic term $\propto\hat{\vec s}\vec B$ in the Hamiltonian remains invariant, see p. [106]29. The rotation matrix $R\in\mathrm{SO}(3)$ in the usual 3D space can be straightforwardly generated from the rotation matrix $U\in\mathrm{SU}(2)$ in the spinor space:[14]

$$\vec r\,'=R\vec r,\quad \xi'=U\xi,\quad (R)_{ij}=\frac{1}{2}\mathrm{Sp}(\sigma_iU\sigma_jU^\dagger), \tag{13.39}$$

(where ξ is a spinor) which uniquely defines rotation operator in real space R based on given rotation operator in spinor space U. The inverse relation is not unique: for a given R, both U and $-U$ are o.k., see p. [106]30. Using ([106]2.23), one obtains the inverse of (13.39) for the special case (13.38):

$$\pm U_\varphi=\sigma_0\cos\frac{\varphi}{2}+i\sigma_3\sin\frac{\varphi}{2},\quad U_\varphi^{-1}=U_\varphi^\dagger,\quad (R_\varphi\vec\sigma)_i=U_\varphi^\dagger\sigma_iU_\varphi,\quad i=1,2,3, \tag{13.40}$$

where I've assumed the rotation in the (x,y)-plane with the rotation matrix R_φ defined in (13.38). According to [104][15] the sign "+" is standard in (13.40), see ([104]1.5.33). The last relation in (13.40) is related with two possible ways of transforming a spinful Hamiltonian like (6.1) together with the rotation of the CS. At first, I can consider the Hamiltonian in the initial CS as a function of spin operator(s): $H_1(\vec p,\vec r,\vec s)$, and rotate the spin operator just like any other vector:

$$H_1(\vec p,\vec r,\vec s)=H_1(R_{-\varphi}\vec p\,',R_{-\varphi}\vec r\,',R_{-\varphi}\vec s\,')\stackrel{\text{df}}{=}H_2(\vec p\,',\vec r\,',\vec s\,'). \tag{13.41}$$

Or instead I could consider the Hamiltonian as a 2×2-matrix [using the last identity from (13.40)]

$$H_1(\vec p,\vec r)\longrightarrow U_\varphi H_1(R_{-\varphi}\vec p\,',R_{-\varphi}\vec r\,')U_\varphi^\dagger\stackrel{\text{df}}{=}H_2(\vec p\,',\vec r\,'),\quad \psi_2(\vec r)=U_\varphi\psi_1(R_\varphi^{-1}\vec r). \tag{13.42}$$

The last way is better[16], because it is compatible with the spinor theory [106].

$$\forall\varphi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\quad \cos^2\frac{\varphi}{2}=\frac{1}{2}\left[1+\frac{1}{\sqrt{1+\mathrm{tg}^2\,\varphi}}\right],\quad \sin^2\frac{\varphi}{2}=\frac{1}{2}\left[1-\frac{1}{\sqrt{1+\mathrm{tg}^2\,\varphi}}\right].$$

[12] Some other cool identities for Pauli matrices can be found in the appendix of [103]. See also [104].

[13] We follow the s.c. "passive" (see p. [106]21) point of view for rotations: the CS is (counterclockwise) rotated, and, consequently, coordinates of all particles are changed:

$$\vec r\to\vec r\,'=R\vec r,\quad \vec r\to\vec r\,'=R\vec r,\quad \vec r\to\vec p\,'=R\vec p,\quad \vec p\to\vec p\,'=R\vec p.$$

[14] Красиво это объяснено на стр. [107]38-39. См. также про спинорные представления стр. [107]181-182.

[15] Note that sign $\Im U_\varphi$ in (13.40) is the opposite from the one in ([104]1.5.33), despite that, apparently, also [104] rotates the SC just like me. I have checked that (13.40) coincides with (13.39). However, (13.40) coincides with ([60]58.2) which is nice.

[16] I wonder how this coincides with ([104]1.5.28).

$$\sqrt{1+i\alpha}-\sqrt{1-i\alpha}=i\sqrt{2}\sqrt{\sqrt{1+\alpha^2}-1}.$$

$$\operatorname{arctg} x=\frac{1}{2i}\ln\frac{1+ix}{1-ix}. \tag{13.43}$$

13.1 О линеаризации спектра

См. в промежутке между ([50]163-164), а также между ([17]34-35): ошибочность утверждения $p^2/(2m)-\mu\approx v_{\mathrm{F}}\vec{n}\vec{p}\equiv v_{\mathrm{F}}p$ очевидна при $p=p_{\mathrm{F}}$ и $\mu=p_{\mathrm{F}}^2/(2m)$. В частном случае

$$G_0(\vec{r})=\int\frac{\mathrm{d}^dp}{(2\pi)^d}f(\Omega)\exp[i\vec{p}\vec{r}]\,\delta(p-p_{\mathrm{F}})\Longrightarrow-\delta G_0(\vec{r})=p_{\mathrm{F}}^2G_0(\vec{r}). \tag{13.44}$$

В более общем случае мы предполагаем, что δ-функция в (13.44) имеет конечную ширину $\sim 1/l$, так что (13.44) выполняется лишь приближённо; будем считать его приближением нулевого порядка

$$-\delta G_0(\vec{r})\approx\left[-\delta G_0(\vec{r})\right]^{(0)}\equiv p_{\mathrm{F}}^2G_0(\vec{r}). \tag{13.45}$$

Приближение (13.45) для нас является слишком грубым; давайте найдём поправку следующего (первого) порядка:

$$\left[-\delta G_0(\vec{r})\right]^{(1)}\approx-\delta G_0(\vec{r})-\left[-\delta G_0(\vec{r})\right]^{(0)}=\int\frac{\mathrm{d}^dp}{(2\pi)^d}\left(p^2-p_{\mathrm{F}}^2\right)G_0(\vec{p})\exp[i\vec{p}\vec{r}]\approx 2mv_{\mathrm{F}}\int\frac{\mathrm{d}^dp}{(2\pi)^d}(p-p_{\mathrm{F}})G_0(\vec{p})\exp[i\vec{p}\vec{r}]\Longrightarrow$$

$$\Longrightarrow-\delta G_0(\vec{r})\approx\left[-\delta G_0(\vec{r})\right]^{(0)}+\left[-\delta G_0(\vec{r})\right]^{(1)}=p_{\mathrm{F}}\int\frac{\mathrm{d}^dp}{(2\pi)^d}[2p-p_{\mathrm{F}}]G_0(\vec{p})\exp[i\vec{p}\vec{r}],\quad\Longrightarrow\frac{p^2}{2m}\approx v_{\mathrm{F}}\left(p-\frac{p_{\mathrm{F}}}{2}\right). \tag{13.46}$$

Т.к. я не умею записывать оператор p (модуль импульса) в координатном пространстве, использование приближения (13.46) фактически обрекает меня на проведение всех последующих вычислений в импульсном пространстве.

13.2 Asymptotics of small-momenta integrals

Based on sec. 3.6, one of important assumptions of the loop expansion is that integrals by CD-momenta converge on the scale $ql\lesssim x\ll 1$, where, x is some small parameter. E.g., $x=l/L$, where L is the system size (e.g., ring's circumference in sec. 10.2 and 11), or $x=\sqrt{x_a^2+x_b^2}$ – the SOI amplitude from sec. 6. However, our expressions can be sometimes splited into "normal" and "defective" parts, where CD-integrals converge on the scale $ql\lesssim x$ and $ql\lesssim 1$ correspondingly. These "defective" parts must be somehow cancelled, otherwise the theory becomes inconsistent. Let us study some examples (everywhere $ql=xQ$):

$$x\int_0^\infty\frac{\mathrm{d}(ql)}{x^2+q^2l^2+q^4l^4}=\int_0^\infty\frac{\mathrm{d}Q}{1+Q^2}-x\int_0^\infty\frac{\mathrm{d}(ql)}{1+q^2l^2},$$

Note that the "normal" term is always the leading one. It will be the same, e.g., in $x\int_0^\infty\frac{\mathrm{d}(ql)}{x^2+q^2l^2(1+x^2)+q^4l^4}$.

Let $h(kl)$ be an expression for some GFB, e.g., for a HB. We want to calculate the asymptotic expansion for several integrals, and the first one is

$$I_{0n}(x)=\int_0^\infty\mathrm{d}k\frac{h_n(kl)}{x^2+k^2l^2},\quad x\ll 1\Longrightarrow I_{0n}(x)=h_n(0)\int_0^\infty\mathrm{d}k\frac{1}{x^2+k^2l^2}+I_n,$$

where

$$I_n(x)=\int_0^\infty\mathrm{d}k\frac{h_n(kl)-h_n(0)}{x^2+k^2l^2},\quad x\ll 1,\quad h_n(kl)\approx h_n(0)+c(kl)^n,\quad kl\ll 1,\quad c=\mathrm{const}\neq 0.$$

In reality we are interested only in $n\leq 2$. We can not just expand the $h_n(kl)$ in the integrand in Taylor series since it spoils the convergence. The case $n=1$ is somewhat similar to the WL: in the large interval $x/l\ll k\ll 1/l$ the integrand is $\sim 1/k$, that is,

$$I_1(x)\approx\int_{x/l}^{1/l}\frac{c}{kl}=-\frac{c}{l}\log x.$$

Next,

$$I_2(x) = \int_0^\infty \mathrm{d}k g(kl) - x^2 \int_0^\infty \mathrm{d}k \frac{g(kl)}{x^2 + k^2 l^2}, \quad g(kl) \stackrel{\mathrm{df}}{=} \frac{h_n(kl) - h_n(0)}{(kl)^n}, \quad g(kl \ll 1) \approx c \neq 0.$$

The first integral converges on $lk \sim 1$, while the second one converges on $lk \sim x \ll 1$. We see that the second integral is basically $I_{0n}(x)$ with new $h_n(kl) = g(kl)$. The only difficulty might be calculating $\int_0^\infty \mathrm{d}k g(kl)$, however, it is x-independent, so might be not important. Following this prescription, one can build asymptotic of arbitrary length for a typical integral over a small CD momenta.

13.2.1 The same using dimensionless variables

In problems with SOI, we have a mass $x \ll 1$ defined in (6.3). It makes sense to introduce the dimensionless variable $Q = lq/x$, so that $\int \mathrm{d}Q$ converges on the scale $Q \sim 1$. Now let us assume that $h - h_0 \sim c(kl)^2 = c(xK)^2$ and introduce $g(K)$, so that

$$\int_0^\infty \frac{h - h_0}{1 + K^2} \mathrm{d}K = cx \int_0^\infty \frac{h - h_0}{c(lk)^2} \mathrm{d}(lk) - cx^2 \int_0^\infty \frac{\mathrm{d}K}{1 + K^2} g(K), \quad g(K) \stackrel{\mathrm{df}}{=} \frac{h(K) - h_0}{c(xK)^2}, \quad g(0) = 1.$$

The first integral has no convergence scale, or, from the numerical point of view, it converges on $lq \sim 1$. The second integral has usual "diffusive" convergence on the scale $K \sim 1$. Now let us generalize:

$$\int_0^\infty \frac{C(X,Y)}{Z(K,Q)} \mathrm{d}K\mathrm{d}Q = \int_0^\infty g(K,Q)\mathrm{d}K\mathrm{d}Q - \int_0^\infty \frac{\mathrm{d}K\mathrm{d}Q}{Z(K,Q)} g(K,Q), \quad g(K,Q) \stackrel{\mathrm{df}}{=} \frac{C(X,Y)}{Z(K,Q) - 1}, \tag{13.47}$$

and we supposed that $g(K,Q)$ is a regular function in the neighborhood of the point $(K = 0, Q = 0)$. One can ask a question: Who is Who in (13.47)?  For the moment this is all imprecise, living example required! Let us answer on the example of sec. 8.4: $C(X,Y) = x^2(X,Y)^2$ comes the product of left and right HBs, and $Z(K,Q)$ comes from the product of three diffusons. [K and Q are dimensionless variables; each HB is divided by x, and each diffuson is multiplied by x^2, so that $Z(0,0) = 1 + (x_a, x_b)^2$.] More precisely, $C(X,Y)$ comes from $[LR - (LR\big|_{K=Q=0})]/x^2$. Here I say "comes from" instead of "equal" because (13.47) appears after the angular integration, when all integrals are done except for $\int_0^\infty \mathrm{d}K\mathrm{d}Q$. I think that $g(0,0)\big|_{x=0}$ =const. I think, that the SOI-dependent part of (13.47) can be calculated up to the arbitrary order of (x_a, x_b): in fact, based on the родство(=familiarity?) of Z with the quantity Y form Sec. 6.8, we assume that $Z(K,Q) - 1 = (xa, xb)^2(K,Q)^2(1 + \tilde{Z}(K,Q))$, so for both terms in the lhs of (13.47) we can define new quantities $\tilde{C}$ and $\tilde{Z}$ and then repeat the whole procedure for them.

Chapter 14

Density matrix with spin

Following the statistical approach [108], the complete density matrix of a many-particle system is defined as an average over an ensemble of the systems. Each system in the ensemble is identical in the sence that it has the same Hamiltonian, number of particles, size, etc. The number of systems in the ensemble is infinite. Every single system is characterized by its wavefunction ψ_α. Introducing the probability (a usual statistical one) P_α, we define the complete density matrix as

$$\langle\lambda|\rho|\lambda'\rangle = \int \psi(\lambda)\psi^\dagger(\lambda')P[\psi]\mathcal{D}[\psi],$$

where the functional integration is restricted to normalized WF ψ. The true (=explicit) sence of the functional integral comes out after we express ψ_α via the linear combination of the eigenfunctions φ_n:

$$\langle\lambda|\rho|\lambda'\rangle = \int \mathrm{d}\alpha_1\mathrm{d}\alpha_2\ldots\mathrm{d}\beta_1\mathrm{d}\beta_2\ldots P[\{\alpha_n\}_{n\geq1},\{\beta_m\}_{m\geq1}]\sum_{m,n\geq1}\alpha_n\beta_m^*\varphi_n(\lambda)\varphi_m^\dagger(\lambda'),$$

which I was writing before as

$$\langle\lambda|\rho|\lambda'\rangle = \sum_{nm} W_{nm}\varphi_n(\lambda)\varphi_m^*(\lambda'), \quad W_{nm} = \sum_\alpha P_\alpha C_m^\alpha C_n^{\alpha*}, \quad \psi_\alpha(\lambda) = \sum_n C_n^\alpha\varphi_n(\lambda). \tag{14.1}$$

The off-diagonal elements W_{nm} in (14.1) would oscillate like $\cos(E_n - E_m)t$. Consider the case when W_{nm} is a slowly varying function of n,m. Then a sum of all these cosines will be a function, that decays in time on a characteristic scale of the mean level spacing $\delta \sim v/L^d$, where d is the dimension. We usually have $d = 2$; then L^2 is just a surface area. So, for a large system on a large time scale, only diagonal elements survive in (14.1):

$$\langle\lambda|\rho|\lambda'\rangle \approx \sum_n W_{nn}\varphi_n(\lambda)\varphi_n^*(\lambda'). \tag{14.2}$$

Another case when (14.2) is correct, is equilibrium. In fact, the expectation of a time-independent quantity, calculated with (14.2), will be time independent, like this should be in equilibrium, when all the relaxation procecces in the system are finished. Then $W_n n = \exp[-E_n/T]/Z$, where E_n is the eigenenergy and Z is the statistical sum.

Now let us consider the opposite situation in (14.1), when $W_{nm} \propto \delta_{n,m} + \delta_{n,m\pm l}$, or δ (according to `estimates.nb`, for a mesoscopic ring $1/\delta \sim 10^{-7}\mathrm{sec} \gg \tau \sim 10^{-14}\mathrm{sec}$) is large (like e.g. in a quantum dot). Then off-diagonal term gives us single-standing oscillating contribution, and we have to live with it. Then (14.2) may be not a nice approximation.

In conclusion I suspect that (14.2) is strictly true only in equilibrium; out of equilibrium it is an approximation which can be used for times $\gtrsim 1/\delta$. Thus it should work, e.g., for an ordinary-size (not small) piece of metal, when current flows, so that it is not in equilibrium. This brings me back to an argument of VK, who claimed that our non-equilibrium current from sec. 11. BTW, a nice article on density matrix, recomended by [19] and available online: [108].

In [19], (14.2)=([19]1.1) is claimed to be true in the most general case of an arbitrary system. However, I suspect that the authors implied equilibrium.

14.1 In the first quantization

In equilibrium[1] we use the expression (14.2), ([19]1.1) for the density matrix of a N-particle system

$$\left\langle \vec{\lambda}|\rho_N|\vec{\lambda}'\right\rangle = \sum_{\vec{n}}{}' W_{\vec{n}}\psi_{\vec{n}}(\vec{\lambda})\psi^*_{\vec{n}}(\vec{\lambda}'), \quad \sum_{\vec{n}}{}' W_{\vec{n}} = 1, \quad \vec{\lambda} \equiv (\lambda_1 \dots \lambda_N), \tag{14.3}$$

$$\hat{H}(\vec{\lambda})\psi_{\vec{n}}(\vec{\lambda}) = E_{\vec{n}}\psi_{\vec{n}}(\vec{\lambda}), \quad \sum_{\vec{\lambda}} \psi^*_{\vec{n}}(\vec{\lambda})\psi_{\vec{n}'}(\vec{\lambda}) = \delta_{\vec{n}\vec{n}'}, \tag{14.4}$$

where $\sum'_{\vec{n}}$ denotes the sum over all *distinct* N-particle states. We will assume that $\lambda_i \equiv (\vec{r}_i, d_i)$, where $d_i \in (\uparrow, \downarrow)$ is the z-spin projection of an electron; that is, we have $\int \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N$ and $\delta(\vec{n} - \vec{n}')$ instead of $\sum_{\vec{\lambda}}$ and $\delta_{\vec{n}\vec{n}'}$ in (14.4).

In the absence of interaction,

$$\hat{H}(\vec{\lambda}) = \sum_{i=1}^{N} h(\lambda_i), \quad \psi_{\vec{n}}(\vec{\lambda}) = \frac{1}{\sqrt{N!}} \sum_{\tau \in S_N} \varphi_{\tau_{n_1}}(\lambda_1) \dots \varphi_{\tau_{n_N}}(\lambda_N) \operatorname{sign} \tau,$$

$$h(\lambda)\varphi_m(\lambda) = \varepsilon_m \varphi_m(\lambda), \quad \sum_{\lambda} \varphi^*_m(\lambda)\varphi_{m'}(\lambda) = \delta_{mm'} \quad \sum_{\vec{n}}{}' = \sum_{n_1 < \dots < n_N},$$

where S_N is the perμtation group of order N. Then an expectation value of a single-particle operator is

$$A_{nm} = \sum_{d,d'=\uparrow,\downarrow} \int \mathrm{d}^d r \varphi_n(\vec{r}, d)\langle d|\hat{A}(\vec{r})|d'\rangle\varphi_m(\vec{r}, d'). \tag{14.5}$$

From ([19]1.8a) and (14.17) we get

$$\left\langle \lambda_1|\rho_1|\lambda'_1\right\rangle = N \operatorname*{Sp}_{2\dots N} \left\langle \vec{\lambda}|\rho_N|\vec{\lambda}'\right\rangle = \frac{1}{(N-1)!} \sum_{n_1 < \dots < n_N} W_{\vec{n}} \sum_{\tau,\tau' \in S_N} \varphi_{n_{\tau_1}}(\lambda_1)\varphi^*_{n_{\tau'_1}}(\lambda'_1) \times$$

$$\times \operatorname*{Sp}_{2\dots N} \varphi_{n_{\tau_2}}(\lambda_2)\varphi^*_{n_{\tau'_2}}(\lambda'_2) \dots \varphi_{n_{\tau_N}}(\lambda_N)\varphi^*_{n_{\tau'_N}}(\lambda'_N) \operatorname{sign}\tau \operatorname{sign}\tau' =$$

$$= \frac{1}{(N-1)!} \sum_{n_1 < \dots < n_N} W_{\vec{n}} \sum_{\tau \in S_N} \varphi_{n_{\tau_1}}(\lambda_1)\varphi^*_{n_{\tau_1}}(\lambda'_1) =$$

$$= \sum_{n_1 < \dots < n_N} W_{\vec{n}} \sum_{i=1}^{N} \varphi_{n_i}(\lambda_1)\varphi^*_{n_i}(\lambda'_1) = \sum_{n_1 \neq \dots \neq n_N} \frac{W_{\vec{n}}}{N!} \sum_{i=1}^{N} \varphi_{n_i}(\lambda_1)\varphi^*_{n_i}(\lambda'_1),$$

where we used that

$$\forall i = 2 \dots N \quad \operatorname*{Sp}_i \left[\varphi_{n_{\tau_i}}(\lambda_i)\varphi_{n_{\tau'_i}}(\lambda_i)\right] = \delta_{\tau_i\tau'_i},$$

given that there are no equal numbers among $n_2 \dots n_N$.

Now let us put $N = 2$. Then

$$\sum_n w_n\varphi_n(\lambda)\varphi^*_n(\lambda') = \langle\lambda|\rho_1|\lambda'\rangle = \sum_n \varphi_n(\lambda)\varphi^*_n(\lambda') \sum_{\{n'|n' \neq n\}} W_{(nn')}, \quad W_{(nn')} \equiv W_{\vec{n}}. \tag{14.6}$$

In equilibrium w_n is given by (14.5) with $\hat{A} = \exp[-\beta\hat{H}]/z$:

$$w_n \equiv w_{nn} = \frac{1}{z} \sum_{d,d'=\uparrow,\downarrow} \int \mathrm{d}^d r \varphi_n(\vec{r}, d)\langle d|\exp[-\beta\hat{H}(\vec{r})]|d'\rangle\varphi_n(\vec{r}, d') = \frac{1}{z}\exp[-\beta\varepsilon_n],$$

$$W_{(nn')} = \frac{1}{Z}\exp[-\beta(\varepsilon_n + \varepsilon_{n'})] = \frac{z^2}{Z} w_n w_{n'}, \tag{14.7}$$

where z and Z are statistical sums for ρ_1 and ρ_2. Thus in equilibrium the probability $W_{(nn')}$ is factorized. By writing (14.7) we fixed the value of $W_{(nn)} \neq 0$, which could be arbitrary chosen[2] until now.

[1] See sec. 14

[2] For example, we could postulate $\forall n W_{(nn)} = 0$; this would simplify (14.6).

In order to get the expression for ρ_2, now imagine we have two particles in our non-interacting system: The eigenstate $\varphi_{(nn')}(\vec{r}_1,\vec{r}_2)$ of the two-particle Hamiltonian $H(\vec{r}_1)+H(\vec{r}_2)$ is a four-component wave-function. Its components are[3]

$$\varphi_{(nn')}(\vec{r}_1,\vec{r}_2;\mu) = -\varphi_{(n'n)}(\vec{r}_2,\vec{r}_1;\overline{\mu}) = \frac{1}{\sqrt{2}}\left[\varphi_n(\vec{r}_1,d_1)\varphi_{n'}(\vec{r}_2,d_2) - \varphi_n(\vec{r}_2,d_2)\varphi_{n'}(\vec{r}_1,d_1)\right],$$
$$d,d' \in ud, \quad \mu = d_1d_2, \quad \overline{\mu} = d_2d_1, \quad \mu \in \{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}.$$

According to (14.3)

$$\langle \vec{r}_1\vec{r}_2;\mu|\rho_2|\vec{r}_2\,'\vec{r}_1\,';\nu\rangle = \sum_{n<n'} W_{(nn')}\varphi_{(nn')}(\vec{r}_1,\vec{r}_2;\mu)\varphi^*_{(nn')}(\vec{r}_1\,',\vec{r}_2\,';\nu) = \tag{14.8}$$
$$= \sum_{n,n'} \frac{W_{(nn')}}{2}\varphi_{(nn')}(\vec{r}_1,\vec{r}_2;\mu)\varphi^*_{(nn')}(\vec{r}_1\,',\vec{r}_2\,';\nu), \quad \sum_{n<n'} W_{(nn')} = 1, \quad \mu,\nu \in \{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}.$$

$$\frac{Z}{z^2}\langle \vec{r}_1\vec{r}_2;\mu|\rho_2|\vec{r}_2\,'\vec{r}_1\,';\nu\rangle = \langle \vec{r}_1,d_1|\rho_1|\vec{r}_1\,',d_1'\rangle\langle \vec{r}_2,d_2|\rho_1|\vec{r}_2\,',d_2'\rangle - \tag{14.9}$$
$$-\langle \vec{r}_1,d_1|\rho_1|\vec{r}_2\,',d_2'\rangle\langle \vec{r}_2,d_2|\rho_1|\vec{r}_1\,',d_1'\rangle, \quad \mu = d_1d_2, \quad \nu = d_1'd_2', \quad \mu,\nu \in \{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}.$$

This coincides(check it!) with the result ([108]5.30) for a system composed of N non-interacting electrons[4]:  I don't trust this result completely, since I suspect [108] to use a different normalization; ask DL how to prove it in SQ.

$$N(N-1)\langle \vec{r}_1\vec{r}_2;\mu|\rho_2|\vec{r}_2\,'\vec{r}_1\,';\nu\rangle = \langle \vec{r}_1,d_1|\rho_1|\vec{r}_1\,',d_1'\rangle\langle \vec{r}_2,d_2|\rho_1|\vec{r}_2\,',d_2'\rangle - \tag{14.10}$$
$$-\langle \vec{r}_1,d_1|\rho_1|\vec{r}_2\,',d_2'\rangle\langle \vec{r}_2,d_2|\rho_1|\vec{r}_1\,',d_1'\rangle, \quad \mu = d_1d_2, \quad \nu = d_1'd_2', \quad \mu,\nu \in \{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}.$$

Note the similarity with the analogous expression (which follows from Wick's theorem) for the two-particle Green function ([5]10.14). However, (14.10) is not the same, as Wick's theorem[5]. An idea for the future: the measure of entanglement should be similar to that of polarization of light in [108]. Also in [109] the spin polarization is discussed.

Let us now change spin basis in (14.8) by introducing new states: Its basis is composed by four second-rank spinors (each has 4 components, see [60], §56, p. 253)

$$\psi^S_{(nn')}(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}\left[\varphi_{(nn')}(\vec{r}_1\uparrow,\vec{r}_2\downarrow) - \varphi_{(nn')}(\vec{r}_1\downarrow,\vec{r}_2\uparrow)\right],$$
$$\psi^{T^0}_{(nn')}(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}\left[\varphi_{(nn')}(\vec{r}_1\uparrow,\vec{r}_2\downarrow) + \varphi_{(nn')}(\vec{r}_1\downarrow,\vec{r}_2\uparrow)\right], \tag{14.11}$$
$$\psi^{T^-}_{(nn')}(\vec{r}_1,\vec{r}_2) = \varphi_{(nn')}(\vec{r}_1\downarrow,\vec{r}_2\downarrow), \quad \psi^{T^+}_{(nn')}(\vec{r}_1,\vec{r}_2) = \varphi_{(nn')}(\vec{r}_1\uparrow,\vec{r}_2\uparrow), \tag{14.12}$$

so that

$$\varphi_{(nn')}(\vec{r}_1\downarrow,\vec{r}_2\uparrow) = \frac{1}{\sqrt{2}}\left[\psi^{T^0}_{(nn')}(\vec{r}_1,\vec{r}_2) - \psi^S_{(nn')}(\vec{r}_1,\vec{r}_2)\right],$$
$$\varphi_{(nn')}(\vec{r}_1\uparrow,\vec{r}_2\downarrow) = \frac{1}{\sqrt{2}}\left[\psi^{T^0}_{(nn')}(\vec{r}_1,\vec{r}_2) + \psi^S_{(nn')}(\vec{r}_1,\vec{r}_2)\right]. \tag{14.13}$$

Using (14.12) and (14.13), we express $\varphi_{(nn')}$ in terms of $\psi^\alpha_{(nn')}$. We introduce

$$\{\varphi^\mu\}^3_{\alpha=0} = \{\downarrow\downarrow,\downarrow\uparrow,\uparrow\downarrow,\uparrow\uparrow\}, \quad \{\psi^\alpha\}^3_{\mu=0} = \{S,T^-,T^0,T^+\}.$$

[3] By $\mu = d_1d_2$ I mean that, e.g., if $d_1 = \downarrow$ and $d_2 = \uparrow$, then $\mu = \downarrow\uparrow$. The spirit is that of a diadic.
[4] Eq. (14.10) has been checked for the simpliest case of pure state of a two-particle system.
[5] This is better seen in the SQ representation of the DM, see sec. 14.2. Then (14.10) is just a $T \neq 0$ Wick's theorem which is proven on p. [51]76 using path integration.

Now let us rewrite (14.11) and (14.12) in matrix notation:

$$\begin{array}{cc} \psi^\alpha = A_{\alpha\beta}\varphi^\beta, & \varphi^\mu = A^{-1}_{\mu\alpha}\psi^\alpha, \\ A^{-1} = A^T, & \det A = 1, \end{array} \qquad A = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14.14}$$

Let us denorte with ρ_{ST} matrix components of ρ_2 in the $\{S, T^-, T^0, T^+\}$ basis. One obtains ρ_{ST} by substituting (14.14) into the right-hand side of (14.8):

$$\langle \vec{r}_1, \vec{r}_2; \mu|\rho_2|\vec{r}_1\,', \vec{r}_2\,'; \nu\rangle = \sum_{\alpha,\beta} A^{-1}_{\mu\alpha} A^{-1}_{\nu\beta} \langle \vec{r}_1, \vec{r}_2; \alpha|\rho_{\mathrm{ST}}|\vec{r}_1\,', \vec{r}_2\,'; \beta\rangle,$$

$$\langle \vec{r}_1, \vec{r}_2; \alpha|\rho_{\mathrm{ST}}|\vec{r}_1\,', \vec{r}_2\,'; \beta\rangle = \sum_{\mu,\nu} A_{\alpha\mu} A_{\beta\nu} \langle \vec{r}_1, \vec{r}_2; \mu|\rho_2|\vec{r}_1\,', \vec{r}_2\,'; \nu\rangle,$$

$$\alpha, \beta \in \{S, T^-, T^0, T^+\}, \quad \mu, \nu \in \{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}.$$

In particular, for the singlet we get (omitting coordinates)

$$2\langle S|\rho_{\mathrm{ST}}|S\rangle = \langle\downarrow\uparrow|\rho_2|\downarrow\uparrow\rangle + \langle\uparrow\downarrow|\rho_2|\uparrow\downarrow\rangle - \langle\downarrow\uparrow|\rho_2|\uparrow\downarrow\rangle - \langle\uparrow\downarrow|\rho_2|\downarrow\uparrow\rangle. \tag{14.15}$$

One can consider ρ_2 as a 4×4 matrix in its spin indices. Then it is possible to rewrite (14.15) in the nice form $\langle S|\rho_2|S\rangle = \mathrm{Sp}[\rho_2\hat{S}]$, where the 4×4 matrix $\hat{S}$ it is natural to call "singlet density". Analogously this could be done for triplet components.

We are interested in the the situation, when singlet (or triplet) is formed by two electrons with coordinates $\vec{r}_1$ and $\vec{r}_2$. That is, our density matrix is *diagonal* in coordinate space:

$$\langle\lambda_1\lambda_2|\rho_2|\lambda_2'\lambda_1'\rangle = \langle a^*_{\lambda_1'} a^*_{\lambda_2'} a_{\lambda_2} a_{\lambda_1}\rangle = \langle \vec{r}_1, \vec{r}_2; \mu|\rho_2|\vec{r}_1, \vec{r}_2; \nu\rangle, \quad \mu, \nu \in \{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}.$$

Thus the first term in (14.10) is just density-density correlations, while the diagram for the second term is just a bubble (like Drude conductivity, only with different vertices).

One may ask: Why had I no term, analogous to the second term in (14.10), when I was calculating curren-current correlator in my thesis [88]? Apparently, current-current correlator should be calculated with the same diagrams, as the density-density correlator, and the latter seems to be almost the same as our 2-particle density matrix... The answer: however, they are not the same. See the note about different averages on p. 75.

14.2 In SQ

I want to erase the previous section. The only problem I have, is to express ρ^2 via ρ^1 in the SQ formalism. It looks very much like Wick theorem...

Since there is no clear standart in definition of the density matrix (e.g. about its normalization), let me write it from scratch. One- and twoparticle operators are written for the N-particle system as ([60]64.1) and ([60]64.14):

$$A_N(\vec{r}_1 \dots \vec{r}_N) = \sum_{1\le a\le N} A_1(\vec{r}_a), \quad B_N(\vec{r}_1 \dots \vec{r}_N) = \sum_{1\le a<b\le N} B_2(\vec{r}_a, \vec{r}_b). \tag{14.16}$$

Their average values are calculated through one- and twoparticle density matrices ρ_1 and ρ_2 as[6]

$$\begin{aligned} \overline{A} = \mathrm{Sp}[\rho_N A_N] = N\,\mathrm{Sp}[\rho_N A_1] \overset{\mathrm{df}}{=} \mathrm{Sp}[\rho_1 A_1] &\Longrightarrow \rho_1 = N \operatorname*{Sp}_{2\dots N} \rho_N, \\ \overline{B} = \mathrm{Sp}[\rho_N B_N] = \frac{N(N-1)}{2}\,\mathrm{Sp}[\rho_N B_2] \overset{\mathrm{df}}{=} \mathrm{Sp}[\rho_2 B_2] &\Longrightarrow \rho_2 = \frac{N(N-1)}{2} \operatorname*{Sp}_{3\dots N} \rho_N. \end{aligned} \tag{14.17}$$

Already from here we get $\mathrm{Sp}\,\rho_1$ and $\mathrm{Sp}\,\rho_2$, but let us also calculate it in another way:

$$\mathrm{Sp}\,\rho_1 = \sum_{\lambda\lambda'} \langle\lambda|\rho_1|\lambda'\rangle\delta_{\lambda'\lambda} = \mathrm{Sp}[\rho_1\hat{o}_1], \quad \delta_{\lambda\lambda'} = \langle\lambda|\hat{o}|\lambda'\rangle \Longrightarrow \hat{o}_1 = \mathbb{1} \Longrightarrow o_N(\vec{r}_1 \dots \vec{r}_N) = \sum_{a=1}^{N} 1 = N.$$

[6] Note that 1/2 from ([19]1.10) is missing in (14.17). I and Sigi suspect that this 1/2 is taken from ([60]64.15) – that is, from the expression for B in the upper line of (14.17); see also p. [51]18 where ([60]64.15) is properly explained. However, I can not rely on [19] since (i) I've found important mistake there (see p. **??**), and (ii) they use the normalization $\mathrm{Sp}\,\rho = 1$, while I need $\mathrm{Sp}\,\rho_1 = N$.

Thus $\hat{o}$ is just an operator of the number of particles. This means its average (for a system with fixed N) is[7]

$$\mathrm{Sp}[\rho_N o_N] = N\,\mathrm{Sp}[\rho_1 o_1] = N, \quad \text{so that} \quad \mathrm{Sp}\,\rho_1 = N. \tag{14.18}$$

Analogously, for ρ_2 we also can introduce another operator o:

$$\mathrm{Sp}\,\rho_2 = \sum_{\lambda_1\lambda_1'\lambda_2\lambda_2'} \langle\lambda_1\lambda_2|\rho_2|\lambda_2'\lambda_1'\rangle\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} = \mathrm{Sp}[\rho_2\hat{o}_2], \quad \delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2} = \langle\lambda_1\lambda_2|\hat{o}_2|\lambda_2'\lambda_1'\rangle \Longrightarrow$$
$$\Longrightarrow \hat{o} = \mathbb{1} \Longrightarrow o_N(\vec{r}_1\ldots\vec{r}_N) = \sum_{1\le a<b\le N} 1 = \frac{N(N-1)}{2} \Longrightarrow \mathrm{Sp}\,\rho_2 = \frac{N(N-1)}{2}. \tag{14.19}$$

When calculating "mixed" averages like $\overline{A+B}$, where A and B are one- and two-particle operators, we use ρ_2 and

$$\overline{A+B} = \mathrm{Sp}[(A_2+B_2)\rho_2] = \overline{A} + \overline{B},$$

where A_2 is obtained from A_1 using (14.16).

Let us now move to the SQ representation, where operators A and B are given by the same formulas, no matter how many particles are in the system. That's why we ommit indices $1,2,N$ for A and B:

$$A = \sum_{\alpha\beta}\langle\alpha|A_1|\beta\rangle a_\alpha^\dagger a_\beta, \quad B = \frac{1}{2}\sum_{\lambda\mu\nu\kappa}\langle\lambda\mu|B_2|\nu\kappa\rangle a_\lambda^\dagger a_\mu^\dagger a_\nu a_\kappa. \tag{14.20}$$

Like I do it with A and B in (14.20). I can generate N-particle density matrix operator in SQ from its MEs $\langle\lambda_N\ldots\lambda_1|\hat{\rho}_N|\lambda_1\ldots\lambda_N\rangle$ (which I now in the FQ) according to

$$\hat{\rho}_N \overset{\mathrm{df}}{=} \frac{1}{N!}\sum_{\binom{\lambda_1\ldots\lambda_N}{\lambda_1'\ldots\lambda_N'}} \langle\lambda_N\ldots\lambda_1|\hat{\rho}_N|\lambda_1\ldots\lambda_N\rangle a_{\lambda_N}^\dagger\ldots a_{\lambda_1}^\dagger a_{\lambda_1}\ldots a_{\lambda_N}. \tag{14.21}$$

By analogy it would be nice do define DL: (14.22) is wrong!

$$\hat{\rho}_1 \overset{\mathrm{df}}{=} \sum_{\lambda,\lambda'}\langle\lambda|\hat{\rho}_1|\lambda'\rangle a_\lambda^\dagger a_{\lambda'}, \quad \hat{\rho}_2 \overset{\mathrm{df}}{=} \frac{1}{2}\sum_{\binom{\lambda_1\lambda_2}{\lambda_1'\lambda_2'}} \langle\lambda_2\lambda_1|\hat{\rho}_2|\lambda_1\lambda_2\rangle a_{\lambda_2}^\dagger a_{\lambda_1}^\dagger a_{\lambda_1'} a_{\lambda_2'}. \tag{14.22}$$

I can use eqs. (14.21) and (14.22) as definitions of $\hat{\rho}_{1,2,N}$ in the SQ, if I check, keeping in mind (14.20) and (14.24), that the following relations, analogous to (14.17) do hold (assuming that $\hat{O}$ is some N-particle operator):

$$\overline{O} = \mathrm{Sp}[\hat{\rho}_N\hat{O}], \quad \overline{A} = \mathrm{Sp}[\hat{\rho}_N\hat{A}] = \mathrm{Sp}[\hat{\rho}_1\hat{A}], \quad \overline{B} = \mathrm{Sp}[\hat{\rho}_N\hat{B}] = \mathrm{Sp}[\hat{\rho}_2\hat{B}]. \tag{14.23}$$

The symbol Sp in (14.23) stands for the trace in the second qantization, and is not the same, as Sp in (14.17); So let us at first precise what we mean by Sp in SQ. We assume that our states are oredered by energy, so that $\varepsilon_n > \varepsilon_m \Longleftrightarrow n > m$. Then for arbitrary operator $\hat{O}$

$$\mathrm{Sp}\,\hat{O} = \sum_{n_1,n_2,\ldots n_\infty=0}^{1} \langle\ldots n_2,n_1|\hat{O}|n_1,n_2,\ldots\rangle,$$
$$|n_1,n_2,\ldots\rangle \overset{\mathrm{df}}{=} (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\ldots|0_1 0_2\ldots 0_\infty\rangle \overset{\mathrm{df}}{=} (\langle\ldots n_2,n_1|)^\dagger. \tag{14.24}$$

Check (14.23)! In the following, by $\overline{\langle\ldots\rangle}$ we are going to imply not only quantummechanical, but also statistical average, so that $\overline{\langle\ldots\rangle} = \mathrm{Sp}[\rho_N\ldots]$.

$$\langle\alpha|\rho_1|\beta\rangle = \overline{\langle a_\alpha^\dagger a_\beta\rangle}, \quad \langle\lambda\mu|\rho_2|\nu\kappa\rangle = \overline{\langle a_\lambda^\dagger a_\mu^\dagger a_\nu a_\kappa\rangle}, \tag{14.25}$$

[7] The normalization of the DM (14.18) is connected with the normalization of the WF and $G_{\mathrm{R/A}}$. I have a mess about this in these notes: my (standart) $G_{\mathrm{R/A}}$ correspond to the normalization by unity, see the note after (13.30).

The advantage of writing ρ_2 in the SQ form (14.25) is that it shows us straightforward connection with the Keldysh-Green function via (9.4). Also it is easy to write ρ_2 in the singlet-triplet representation. To do it, let us at first introduce two-particle operators (from now on we separate spin indices from spatial coordinates)

$$\left.\begin{array}{c} a_\mu(\vec r_1,\vec r_2)\stackrel{\rm df}{=}a_{d_2}(\vec r_2)a_{d_1}(\vec r_1) \\ \Longleftrightarrow a^\dagger_\mu(\vec r_1,\vec r_2)=a^\dagger_{d_1}(\vec r_1)a^\dagger_{d_2}(\vec r_2)\end{array}\right\},\quad \mu=d_1d_2,\quad d_{12}\in ud,\quad \mu\in\{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}.$$

This last equation maps ρ_2 onto 4×4-matrix (in spin space):

$$\langle\vec r_1,\vec r_2;\mu|\rho|\vec r_2,\vec r_1;\nu\rangle=\overline{\langle a^\dagger_\mu(\vec r_1,\vec r_2)a_\nu(\vec r_1\,',\vec r_2\,')\rangle},\quad \mu,\nu\in\{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}. \tag{14.26}$$

We are interested in the density correlator; that is, we want to take diagonal (in coordinate space) elements of (14.26), which have a symmetry

$$\langle\vec r_1,\vec r_2;\mu|\rho_2|\vec r_2,\vec r_1;\nu\rangle=\langle\vec r_1,\vec r_2;\nu|\rho_2^\dagger|\vec r_2,\vec r_1;\mu\rangle.$$

We call (14.26) the standart representation for ρ_2. By default, when speaking about two-particle density matrix, we assume its form (or representation) (14.26).

Then we define anihilation and creation operators in the singlet-triplet representation:

$$a_S(\vec r_1,\vec r_2)=\frac{1}{\sqrt2}[a_{\uparrow\downarrow}(\vec r_1,\vec r_2)-a_{\downarrow\uparrow}(\vec r_1,\vec r_2)],\quad a_{T^+}(\vec r_1,\vec r_2)=a_{\uparrow\uparrow}(\vec r_1,\vec r_2),$$
$$a_{T^0}(\vec r_1,\vec r_2)=\frac{1}{\sqrt2}[a_{\uparrow\downarrow}(\vec r_1,\vec r_2)+a_{\downarrow\uparrow}(\vec r_1,\vec r_2)],\quad a_{T^-}(\vec r_1,\vec r_2)=a_{\downarrow\downarrow}(\vec r_1,\vec r_2),$$

or, in matrix form,

$$\begin{pmatrix} a_S\\ a_{T^+}\\ a_{T^0}\\ a_{T^-}\end{pmatrix}=\begin{pmatrix} 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0\\ 1 & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} a_{\uparrow\uparrow}\\ a_{\uparrow\downarrow}\\ a_{\downarrow\uparrow}\\ a_{\downarrow\downarrow}\end{pmatrix}. \tag{14.27}$$

For example the singlet density will be $\langle\vec r_1,\vec r_2;S|\rho_{ST}|\vec r_2,\vec r_1;S\rangle=\overline{\langle a^\dagger_S(\vec r_1,\vec r_2)a_S(\vec r_1,\vec r_2)\rangle}$. From the practical point of view, instead of calculating ρ_2 directly in one of these representations, it is more convenient to calculate an object[8]

$$\begin{aligned}\langle\vec r_1,\vec r_2;\sigma_\alpha|\rho_\sigma|\vec r_2,\vec r_1;\sigma_\beta\rangle&\stackrel{\rm df}{=}\sum_{ijkl\in ud}\sigma^{ij}_\alpha\sigma^{kl}_\beta\overline{\langle a^\dagger_j(\vec r_1)a^\dagger_l(\vec r_2)a_k(\vec r_2)a_i(\vec r_1)\rangle}=\\ &=\overline{\langle b^\dagger_\alpha(\vec r_1,\vec r_2)b_\beta(\vec r_1,\vec r_2)\rangle}-\delta(\vec r_1-\vec r_2)\sum_{jk\in ud}[\sigma_\beta\sigma_\alpha]_{kj}\langle a^\dagger_ja_k\rangle,\quad \alpha,\beta=0\ldots3,\\ b^\dagger_\alpha(\vec r_1,\vec r_2)&=\sum_{ij\in ud}\sigma^{ij}_\alpha a^\dagger_j(\vec r_2)a_i(\vec r_1),\quad b_\beta(\vec r_1,\vec r_2)=\sum_{kl\in ud}\sigma^{kl}_\beta a^\dagger_l(\vec r_1)a_k(\vec r_2),\\ a^\dagger_j(\vec r_2)a_i(\vec r_1)&=\sum_\alpha\sigma^{ji}_\alpha b^\dagger_\alpha(\vec r_1,\vec r_2),\quad a^\dagger_l(\vec r_1)a_k(\vec r_2)=\sum_\beta\sigma^{lk}_\beta b_\beta(\vec r_1,\vec r_2)\end{aligned} \tag{14.28}$$

Let us call ρ_σ "two-particle density matrix in σ-representation". The inverse to (14.28) transformation[9] one gets with the help of (13.37):

$$\langle\mu|\rho_2|\nu\rangle=\frac14\sum_{\alpha,\beta=0}^{3}\sigma^{d_1d_1'}_\alpha\sigma^{d_2d_2'}_\beta\langle\sigma_\alpha|\rho_\sigma|\sigma_\beta\rangle,\quad \mu=d_1d_2,\quad \nu=d_1'd_2',\quad \mu,\nu\in\{\uparrow\uparrow,\uparrow\downarrow,\downarrow\uparrow,\downarrow\downarrow\}. \tag{14.29}$$

where we've omited coordinates.

Both ρ in (14.26) and ρ^σ in (14.28) are 4×4 matrices. Each element of ρ^σ is a linear combination of four elements of ρ in (14.26). Thus the complete linear transformation between ρ in (14.26) and ρ^σ may contain $16\times4=64$ non-zero coefficients[10]. The number of non-zero coefficients may be further reduced by the above-mentioned symmetry of ρ; In

[8] I've defined operators b in (14.28) hoping to obtain smth like (14.27); however finally I don't see any use for them.
[9] Note similarity between (14.29) and (6.27).
[10] Thus my initial idea, that one could somehow write this transformation with a 4×4 matrix was wrong.

the best case we could get rid of another $16 \times 2 = 32$ coefficients. In reality this number is lower, because in case of diagonal elements of ρ this symmetry does not lead to the reduction of the number of coefficients in the transformation.

At first I wanted to derive direct transformation between ρ_σ and ρ_{ST}; something clear and simple, like (14.27). However, I had to leave my hopes to do it. Instead, at first I calculate ρ^σ; then with the help of *Mathematica* transform it into ρ in the form (14.26). I will use (14.10), checking its coefficient from the comparison with the normalization (14.17). At last, I come to ρ^{ST} using (14.27).

Bibliography

[1] Gilles Montambaux. Coherence and interactions in diffusive systems. originally – cond-mat/0611574; a better format is . See also [36]. 5, 26, 72

[2] G. Bergmann. Weak localization in thin-films – a time-of-flight experiment with conduction electrons. *Physics Reports*, 107(1):1–58, 1984. A nice review written in the spirit of my notes. Details of calculations are shown. Spin effects are also considered. . 5

[3] A. G. Aronov and Yu. V. Sharvin. Magnetic flux effects in disordered conductors. *Rev. Mod. Phys.*, 59(3):755 – 779, Jul 1987. . 5, 112

[4] Всеволод Феликсович Гантмахер. *Электроны в неупорядоченных средах*. Физматлит, 2003. Очень физичная книжка, теория, обильно иллюстрируемая экспериментальными данными, приятно читать! . 5, 25

[5] Алексей Алексеевич Абрикосов, Лев Петрович Горьков, and Игорь Ехильевич Дзялошинский. *Methods of quantum field theory in statistical physics*. Dobrosvet (Moscow), 2nd Russian edition, 1998. Abrikosov, Gor'kov, Dzyaloshinskii, ; Numbering of formulas and, apparently, figures is the same, as in English edition. 5, 6, 8, 12, 20, 21, 23, 26, 27, 29, 45, 51, 59, 78, 79, 81, 112, 113, 120

[6] Thomas Dittrich, Peter Hänggi, Gert-Ludwig Ingold, Bernard Kramer, Gerd Schön, and Wilhelm Zwerger. *Quantum Transport and Dissipation*. Wiley-VCH, Weinheim, 1998. . 5

[7] I. L. Aleiner and A. I. Larkin. Divergence of classical trajectories and weak localization. *Phys. Rev. B*, 54(20):14423–14444, Nov 1996. . 5, 72

[8] Алексей Алексеевич Абрикосов. *Основы теории металлов*. Наука, 1987. A. A. Abrikosov "Fundamentals of the Theory of Metals". 5, 11, 30, 33, 87

[9] Y. M. Galperin. *Introduction to Modern Solid State Physics*. ; the latest version is available on the web. 5, 105

[10] Boris L. Altshuler and A. G. Aronov. Electron-electron interaction in disordered systems. In A. L. Efros and M. Pollak, editors, *Electron-electron interaction in disordered conductors*. Elsevier, 1985. . 5, 10, 46, 53, 92

[11] А. Я. Шик, Л. Г. Бакуева, С. Ф. Мусихин, and С. А. Рыков. *Физика низкоразмерных систем*. Наука Спб, 2001. хороша! A. Ya. Shik et al., Physics of Lowdimensional systems, S-Pb, 2001. 5

[12] Л. Д. Ландау and Е. М. Лифшиц. *Физическая кинетика*, volume 10. Наука, 1979. Landau & Lifshitz X – "Physical kinetics". 5, 79, 82

[13] А. И. Ансельм. *Введение в теорию полупроводников*. М., 1978. 6

[14] И. М. Лифшиц. О структуре энергетического спектра и квантовых состояниях неупорядоченнных конденсированных систем. *УФН*, 83:617, 1964. I. M. Lifshitz, Uspekhi fizicheskih nauk **83**, 617 (1964). 6

[15] Б. И. Шкловский and А. Л. Эфрос. *Электронные свойства легированных полупроводников*. Наука, 1979. Shklovski, Efros, "Electronic properties of doped semiconductors". 6

[16] John H. Davies. *The physics of low-dimensional Semiconductors*. Cambridge Univ. Press, Cambridge, 1997. . 6, 47

[17] Igor V. Lerner. Nonlinear sigma model for normal and superconducting systems: A pedestrian approach. cond-mat/0307471. 6, 38, 42, 78, 84, 116

[18] найти! рассуждения о том, почему в нашей науке не возникает средних от Гриновских функций выше 2 порядка: Мирлин, Горный, 2002; Ефетов, Коган – толстая статья, д.б. выйти где-то в районе 2003. 6

[19] В. Л. Бонч-Бруевич and С. В. Тябликов. *Метод функций Грина в статистической механике*. гос. изд. физ.-мат. лит., Москва, 1962. V. L. Bonch-Bruevich and S. V. Tyablikov, "The Green function method in statistical mechanics". 6, 12, 76, 79, 112, 113, 114, 118, 119, 121

[20] Oleg Chalaev and Daniel Loss. Spin-orbit-induced anisotropic conductivity of a disordered two-dimensional electron gas. *Phys. Rev. B*, 80(3):035305, Jul 2009. arXiv/0902.3277. 6, 20, 27, 29, 31, 47, 48, 50, 54, 56, 57, 64, 65, 67, 68, 69, 71, 72, 79, 90, 92, 107

[21] J■rgen Rammer and H. Smith. Quantum field theoretical methods in transport theory of metals. *Rev. Mod. Phys.*, 58:323, 1986. 7, 10, 20, 22, 32, 33, 76, 78, 79, 80, 81, 82, 87, 96, 103, 110

[22] E Bringuier and K Bhattacharyyal. Hot-electron impact cross section of impurities in semiconductors. *Semiconductor Science and Technology*, 10(8):1065, 1995. 8

[23] J. E. G. Farina. *Quantum theory of scattering processes*. Pergamon Press, 1973. 9

[24] Victor M. Edelstein. Spin polarization of conduction electrons induced by electric-current in 2-dimensional asymmetric electron-systems. *Sol. State Comm.*, 73(3):233–235, January 1990. 10, 50, 56

[25] В. Л. Бонч-Бруевич, И. П. Звягин, Р. Кайпер, А. Г. Миронов, Р. Эндерлайн, and Б. М. Эссер. *Электронная теория неупорядоченных полупроводников*. 1978. 10, 24

[26] Alex Kamenev. Keldysh and Doi-Peliti techniques for out-of-equilibrium systems. cond-mat/0109316. 12, 78, 82, 83, 84

[27] Н. Н. Боголюбов and Д. В. Ширков. *Введение в теорию квантованных полей*. Физматлит Наука, 1993. N. N. Bogoliubov and D. V. Shirkov, "Quantum Fields". Издание 1984 г. – см. 15

[28] Boris L. Altshuler, A. G. Aronov, and D. E. Khmelnitsky. Effects of electron-electron collisions with small energy transfers on quantum localization. *Journal of Physics C: Solid State Physics*, 15:7367–7386, 1982. 16, 105

[29] Henrik Bruus and Karsten Flensberg. *Introduction to Many-body quantum theory in condensed matter physics*. 2 edition, 2002. интересна необычными подходами, но некоторые важные моменты проскакиваются, мол, "очевидно, что. . . ". 20, 24

[30] Victor M. Edelstein. A simple approach to the evaluation of the Hall conductivity in impure metals within the Green function formalism. *JETP Lett.*, 67(2):159–163, January 1998. 20

[31] Vladimir I. Yudson, Evgeni Kanzieper, and Vladimir E. Kravtsov. Limits of the dynamical approach to the nonlinear response of mesoscopic systems. *Phys. Rev. B*, 64:045310, 2001. cond-mat/0012200; Статья, где ВК (и сотоварищи) объясняет, как можно считать проводимость по формуле Ландауера в технике Келдыша. 23

[32] В. Л. Бонч-Бруевич and С. Г. Калашников. *Физика полупроводников*. 23

[33] J. M. Ziman. *Principles of the theory of solids*. Russian ed., 1974. Дж. Займан, Принципы теории твёрдого тела. 23

[34] Shinobu Hikami, Anatoly I. Larkin, and Yosuke Nagaoka. Spin-orbit interaction and magnetoresistance in the two-dimensional random system. *Prog. Theor. Phys.*, 63(2):707–710, Feb 1980. WL correction taking SO scattering (off impurities) and magnetic impurities into account отсканировать. 25, 44

[35] M. A. Skvortsov. Weak antilocalization in a 2D electron gas with chiral splitting of the spectrum. *JETP Lett.*, 67:133, 1998. cond-mat/9712135; an improvement of [73] in that not small values of Rashba $x \gtrsim 1$ are treated. However, Edelstein on p. 7 (after eq. ([73]2.29)) says that he needs $x \ll 1$ in particular, to neglect the last two diagrams on his fig. [73]7 (the ones with the Hikami box). But also apart from this, I think one cannot use a usual approximation ([35]6) in case when $x^2 \gtrsim 1$. 25, 53, 60, 62, 127

[36] Erick Akkermans and Gilles Montambaux. *Mesoscopic Physics of Electrons and Phonons*. Cambridge University Press, New York, 2007. 26, 29, 125

[37] M. Schreiber and M. Ottomeier. Localization of electronic states in 2D disordered systems. *J. Phys.: Condens. Matter*, 4(8):1959–1971, 1992. . 26

[38] Фёдор Максимилианович Куни. *Статистическая физика и термодинамика*. Наука, 1981. . 26, 73, 75, 78, 89, 102, 130

[39] Boris L. Altshuler, Arkadi G. Aronov, and P. Lee. Interaction effects in disordered Fermi systems in 2 dimensions. *Phys. Rev. Lett.*, 44(19):1288–1291, 1980. Correction to the conductivity taking into account interaction. 27, 88

[40] A. M. Finkelshtein. On the frequency and temperature-dependence of the conductivity near a metal-insulator-transition. *JETP Letters*, 37(9):517–520, 1983. calculating correction to the conductivity taking into account interaction; taking spin into account. 27

[41] Boris L. Altshuler, D. E. Khmelnitskii, A. I. Larkin, and P. A. Lee. Magnetoresistance and hall-effect in a disordered 2-dimensional electron-gas. *Phys. Rev. B*, 22(11):5142–5153, 1980. . 27

[42] E. Abrahams, S. V. Kravchenko, and M. P. Sarachik. Colloquium: Metallic behavior and related phenomena in two dimensions. *Rev. Mod. Phys.*, 73(2):251–266, April 2001. Weak localisation correction $\Delta\rho_{xy} = 0$. 27

[43] D. Belitz and T. R. Kirkpatrick. The Anderson-Mott transition. *Rev. Mod. Phys.*, 66:261–380, 1994. . 27

[44] Д. Пайнс. *Элементарные возбуждения в твёрдых телах*. Мир, 1965.
D. Pines, *Elementary excitations in solids*, 1963. 30

[45] P. W. Anderson. *Basic notions of condensed matter physics*. Benjamin/Cummings, 1984. . 30, 73

[46] Н. Ашкрофт and Н. Мермин. *Физика твёрдого тела*, volume 1. Мир, 1979. Ashcroft & Mermin, . 30, 111

[47] У. Харрисон. *Теория твёрдого тела*. Мир, 1970.
W. A. Harrison, *Solid state theory*, 1970. 30

[48] J. M. Ziman. *Electrons and Phonons. The Theory of Transport Phenomena in Solids*. The int. series of monographs on physics. Изд. ин. лит-ры., 1962. Дж. Займан «Электроны и фононы. Явления переноса в твёрдых телах». 33, 66, 87, 94, 102

[49] John Schliemann and Daniel Loss. Anisotropic transport in a two-dimensional electron gas in the presence of spin-orbit coupling. *Phys. Rev. B*, 68:165311, 2003. ; cond-mat/0306528. 33, 50, 66, 67

[50] Alex Kamenev. Many-body theory of non-equilibrium systems. The optimized version is available here . 35, 36, 37, 38, 39, 40, 41, 82, 83, 84, 116

[51] John W. Negele and Henri Orland. *Quantum Many-Particle Systems*. Adelison-Wesley, 1998. . 35, 38, 82, 83, 120, 121

[52] Alex Kamenev and Anton Andreev. Electron-electron interactions in disordered metals: Keldysh formalism. *Phys. Rev. B*, 60:2218 – 2238, 1998. cond-mat/9810191 This is my prefered article about the NLσM. 37, 41, 83, 84

[53] Феликс Александрович Березин. *Метод вторичного квантования*. 1986. . 38

[54] А. М. Цвелик. *Квантовая теория поля в физике конденсированного состояния*. Физматлит, 2002. . 39, 40

[55] Alexander Atland and Ben Simons. Concepts of theoretical solid state physics. , 2001. 39, 40

[56] Y. Oreg, P. W. Brouwer, X. Waintal, and Bertrand I. Halperin. Spin, spin-orbit, and electron-electron interactions in mesoscopic systems. In F. Peeters T. Chakraborty and U. Sivan, editors, *Nano-Physics and Bio-Electronics*. Elsevier. cond-mat/0109541. 44

[57] Oleg Chalaev. Home page: http://quantumtheory.physik.unibas.ch/shalaev/. 44, 129

[58] J■rgen Rammer. *Quantum transport theory*. Perseus books, Reading, Massachusets, 1988. . 44, 114

[59] Emmanuel I. Rashba. Electron spin operation by electric fields: spin dynamics and spin injection. *PHYSICA E*, 20:189–195, January 2004. cond-mat/0309441; see also references 3,18,20,22,23,24 therein. 44

[60] Л. Д. Ландау and Е. М. Лифшиц. *Квантовая механика (нерелятивистская теория)*, volume 3 of *Theoretical physics*. "Наука", 4th Russian edition, 1989. Landau & Lifshitz, "Quantum mechanics – non-relativistic theory". 44, 48, 49, 111, 115, 120, 121

[61] Weng W. Chow and Stephan W. Koch. *Semiconductor laser fundamentals*. Springer, 1999. . 47

[62] Ronald Winkler. *Spin-Orbit Effects in Two-Dimensional Electron and Hole Systems*. Springer, 2003. see p. 55: "often it is reasonable to assume that $E_c = E_v$ and $\{\psi\}_c \equiv \{\psi\}_v$." . 47

[63] Degang Zhang. Exact Landau levels in two-dimensional electron systems with Rashba and Dresselhaus spin-orbit interactions in a perpendicular magnetic field. *J. Phys. A*, 39:L477–L482, 2006. . 47

[64] N. S. Averkiev, L. E. Golub, and M. Willander. Spin relaxation anisotropy in two-dimensional semiconductor systems. *J. Phys.: Condens. Matter*, 14:R271, 2002. . 47

[65] К. Иниксон and Ж. Б. Зюбер. *Квантовая теория поля*, volume 1. Мир, 1984. Claude Itzykson and Jean-Bernard Zuber, "Quantum Field Theory" v.1. 48

[66] Oleg Chalaev and Daniel Loss. Anisotropic conductivity of a disordered 2DEG in the presence of spin-orbit coupling. *Phys. Rev. B*, 77:115352, 2008. arXiv/0708.3504. 48, 64, 65, 67, 72

[67] Ol'ga V. Dimitrova. Vanishing spin-Hall conductivity in 2D disordered Rashba electron gas. cond-mat/0405339, 2004. 48, 56

[68] Zyun Francis Ezawa. *Quantum Hall Effects*. World Scientific, 2000. . 48, 82

[69] S. Y. Liu and X. L. Lei. Vanishing dissipationless spin Hall current in a diffusive two-dimensional electron gas with spin-orbit coupling. unpublished; Авторы - похоже, крутые ребята, впервые встречаю таких, работающих в Китае. См. также их cond-mat/0411629 and 0509365. 50

[70] Naoyuki Sugimoto, Shigeki Onoda, Shuichi Murakami, and Naoto Nagaosa. Intrinsic vs. extrinsic spin Hall effect in the disordered Rashba model. cond-mat/0503475; 21.03.2005 I've sent Murakami an email about it. In Korea he answered me that he agrees with our argument. Our argument also contradicts to his later cond-mat/0504353, where (in particular, in sec. 3) he again makes contradictory claim that the potential of an impurity has to be $\propto \delta(\vec{r} - \vec{r}_a)$. 50, 56

[71] N. A. Sinitsin, E. M. Hankiewicz, W. Teizer, and J. Sinova. Spin-Hall and spin-diagonal conductivity in the presence of Rashba and Dresselhaus spin-orbit coupling. cond-mat/0310315, Oct 2003. 50

[72] John Schliemann and Daniel Loss. Dissipation effects in spin-Hall transport of electrons and holes. *Phys. Rev. B*, 69:165315, 2004. cond-mat/0310108. 51

[73] Victor M. Edelstein. Band-spin-orbit-energy effects in conductivity of two-dimensional weakly disordered semiconductor systems. *J. Phys.: Condens. Matter*, 7:1–18, 1995. Studied magnetoconductivity for the case of small SOI. . 53, 54, 127

[74] Oleg Chalaev and Daniel Loss. Spin-Hall conductivity due to Rashba spin-orbit interaction in disordered systems. *Phys. Rev. B*, 71:245318, 2005. Better read slightly expanded version: cond-mat/0407342. The presentation is available on my home page [57]. 56, 57, 61, 63, 129

[75] J. I. Inoue, G. E. W. Bauer, and L. W. Molenkamp. Suppression of the persistent spin Hall current by defect scattering. *Phys. Rev. B*, 70(4):041303(R), July 2004. cond-mat/0402442 Original paper about $\sigma^z_{yx} = 0$, that provocated us for going into all this spin-Hall adventure [74]. 56

[76] Shuichi Murakami, Naoto Nagaosa, and Shou-Cheng Zhang. Su(2) non-abelian holonomy and dissipationless spin current in semiconductors. *Phys. Rev. B*, 69:235206, 2004. cond-mat/0310005. 56

[77] Shuichi Murakami. Absence of vertex correction for the spin Hall effect in p-type semiconductors. *Phys. Rev. B*, 69:241202, 2004. cond-mat/0405001. 56

[78] John Schliemann and Daniel Loss. Spin-Hall transport of heavy holes in iii-v semiconductor quantum wells. 71:085308, 2005. cond-mat/0405436. 56

[79] Eugene G. Mishchenko, Andrey V. Shytov, and Bertrand I. Halperin. Spin current and polarization in impure 2D electron systems with spin-orbit coupling. *Phys. Rev. Lett.*, 93:226602, 2004. cond-mat/0406730. 56, 57, 58

[80] M. I. D'yakonov and V. I. Perel'. Spin orientation of electrons associated with interband absorption of light in semiconductors. *Zh. Exp. Teor. Fiz.*, 60:1954, 1971. Sov. Phys. JETP **33**, 1053 (1971); Dyakonov-Perel rate of the spin relaxation is defined in eq. (30,31) for $\alpha = 1$. I came to this ref. from cond-mat/0411239. 63

[81] A. Messiah. *Quantum Mechanics*, volume 1. 1958. . 63

[82] Peter Schwab and Roberto Raimondi. Magnetoconductance of a two-dimensional metal in the presence of spin-orbit coupling. *Eur. Phys. J. B*, 25:483–495, 2002. . 64

[83] А. А. Голубенцев. Прямое вычисление проводимости плёнок с магнитными примесями. *письма в ЖЭТФ*, 41(12):527–530, 1985. . 67

[84] D. Vollhardt and P. Wölfle. Diagrammatic, self-consistent treatment of the anderson localization problem in $d \leq 2$ dimensions. *Phys. Rev. B*, 22(10):4666–4679, 1980. see also cond-mat/0007475. 69

[85] Ю. В. Сидоров, М. В. Федорюк, and М. И. Шабунин. *Лекции по теории функций комплексного переменного*. Наука, 3 edition, 1989. . 70

[86] A. A. Logunov, O. A. Hrustalev, and I. A. Kvasnikov, editors. *Selected papers of N. N. Bogoliubov*, chapter "Quasiaverages in statistical mechanics problems". Moscow State University, 1979. in Russian, . 73

[87] Carlo W. J. Beenakker and Christian Schönenberger. Quantum shot noise. *Physics Today*, page 37, May 2003. cond-mat/0605025;jlink. 75

[88] Oleg Chalaev. *Nonequilibrium persistent currents in mesoscopic disordered systems*. PhD thesis, Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy, 2003. The thesis together with the presentation are available on my homepage [57]. 75, 79, 81, 94, 102, 121

[89] Jean Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. Clarendon Press, 3 edition, 1996. . 78

[90] Л. Каданов and Г. Бейм. *Квантовая статистическая механика. Методы функций Грина в теории равновесных и неравновесных процессов*. Frontiers in Physics. Мир, 1988. Leo P. Kadanoff and Gordon Baym, "Quantum statistical mechanics. Green's function Methods in equilibrium and nonequilibrium problems", 1988. 79

[91] Oleg Chalaev and Vladimir E. Kravtsov. Aharonov-Bohm magnetization of mesoscopic rings caused by inelastic relaxation. *Phys. Rev. Lett.*, 89:176601, 2002. cond-mat/0204176. 79, 86, 94, 96, 97

[92] Alexander L. Fetter and John Dirk Walecka. *Quantum theory of many-particle systems*. McGraw-Hill, San Francisco, 1971. 84

[93] R. D. Mattuck. *A guide to Feynman diagrams in the many-body problem*. McGraw-Hill, London, 1976. . 85

[94] Supriyo Datta. *Electronic Transport in Mesoscopic Systems*. Cambridge uni. press, 1997. . 87

[95] Vinay Ambegaokar and Ulrich Eckern. Coherence and persistent currents in mesoscopic rings. *Phys. Rev. Lett.*, 65:381, 1990. 88, 90, 93

[96] Михаил Юрьевич Ковалевский and Сергей Владимирович Пелетминский. *Статистическая механика квантовых жидкостей и кристаллов*. Физ.-мат. лит., 2006. Достойные продолжатели (или вдохновители?) [38] – чётко и красиво объяснены фазовые переходы, квазисредние, квантовые операторы плотностей потоков, ур-ние Навье-Стокса для квантовых жидкостей. 89

[97] Andrey V. Chubukov and Dmitrii L. Maslov. Cooper channel and the singularities in the thermodynamics of a fermi liquid. *Phys. Rev. B*, 76:165111, 2007. Short version: arXiv/0704.3457. 90

[98] K. E. Nagaev. On the shot noise in dirty metal contacts. *Phys. Lett. A*, 169:103–107, 1992. . 94

[99] H. Pothier, S. Guéron, Norman O. Birge, D. Esteve, and M. H. Devoret. Energy distribution function of quasiparticles in mesoscopic wires. *Phys. Rev. Lett.*, 79(18):3490–3493, Nov 1997. 103

[100] В. М. Галицкий and Б. М. Карнаков В. И. Коган. *Задачи по квантовой механике*. Наука, 1981. Galitsky, Karnakov, Kogan. QM problems for students. 111

[101] Barry R. Holstein. *Topics in advanced quantum mechanics*. Addison-Wesley, Redwood City, Calif., 1992. 111

[102] E. N. Economou. *Green's functions in quantum physics*. Berlin, 1983. 111

[103] Victor M. Edelstein. Space dispersion feature of the conduction electron spin resonance in two-dimensional electron systems caused by the absence of 'up-down' symmetry. *J. Phys.: Condens. Matter*, 5(16):2603–2618, April 1993. here you find some cool identities with Pauli matrices. 115

[104] Дмитрий Александрович Варшалович, Анатолий Николаевич Москалёв, and Валерий Кельманович Херсонский. *Квантовая теория углового момента*. Наука, Ленинград, 1975. English translation: Varshalovich, Moskalev, Hersonsky "Quantum theory of the angular momenta". 115

[105] Gabor Zala, Boris N. Narozhny, and Igor L. Aleiner. Interaction corrections at intermediate temperatures: I. longitudinal conductivity and kinetic equation. *Phys. Rev. B*, 64(21):214204, December 2001. cond-mat/0105406; see also Proceedings of the International School of Physics in Varenna, Cource CLI. 115

[106] Л. Биденхарн and Дж. Лаук. *Угловой момент в квантовой физике*, volume 1. 1984. L. C. Biedenharn, J. D. Louck, "Angular momentum in quantum physics", 1981.. 115

[107] Г. Л. Бир and Г. Е. Пикус. *Симметрия и деформационные эффекты в полупроводниках*. Наука, 1972. G. E. Bir and G. L. Pikus, Symmetry and strain-induced effects in semiconductors, Wiley, New York 1974. 115

[108] D. Ter Haar. Theory and applications of the density matrix. *Reports of Progress in Physics*, 24:304–362, January 1961. . 118, 120

[109] К. Блум. *Теория матрицы плотности и её приложения*. Москва, "Мир", 1983. K. Blum, Theory of density matrix and its applications, 1981, Plenum Press, New York. . 120