General information for obtaining the credit points

- one problem per exercise sheet to be handed in
- 50% of total number of points required
- other problems solved together in exercise class

1. Problem: Birthday paradox (10 points)

Calculate the probability that in a set of n people at least two people have birthday on the same day of the year. Now, compare the result with the probability that at least one person in the group has the same birthday as you. Plot both results as a function of n. For which number n do the results exceed probabilities of 50%? Assume $N = 365$ days and equal birthday probabilities for each day.

Hint: First calculate the probability that all n birthdays are different. In the second case start from the probability that all people have a different birthday from yours. Note that for $n > 365$ the probability that there are no people in the group with the same birthday is zero.

2. Problem: Baye's theorem (no points)

Derive the so-called Baye's theorem

$$P(A_i|B) = \frac{P(B|A_i)P(A)}{\sum_j P(B|A_j)P(A_j)} \tag{1}$$

for an arbitrary event B and a partition $\{A_i\}$ of the state space Ω, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_i A_i = \Omega$. Hint: Start from the definition of the conditional probability $P(A_i|B)$ and use the law of total probability $P(B) = \sum_i P(B \cap A_i)$.

Consider as a simple example three boxes A, B, C containing two gold coins, one gold coin and one silver coin and two silver coins, respectively. One box is picked at random and one coin is picked from it at random. Suppose that this coin is gold. What is the probability that the other one is also gold, i.e., that the first box has been chosen.

Hint: If G denotes the event that the first drawn coin is gold, one has to calculate $P(A|G)$.

For your entertainment and/or confusion, you can also have a look at the Boy or Girl paradox (see e.g. `http://en.wikipedia.org/wiki/Boy_or_Girl_paradox`)

3. Problem: Transformation of variables (no points)

3.1. Product of Gaussians with zero average

Let the variable $\hat{x}$ be a Gaussian random number, i.e., distributed according to the probability density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) . \tag{2}$$

Assuming that the average vanishes, i.e., $\mu = 0$, calculate the probability density of the random number $\hat{y} = \hat{x}^2$.

Write the result in the form of the so-called Gamma distribution

$$p(x) = \begin{cases} \dfrac{a^\nu}{\Gamma(\nu)}\, x^{\nu-1}\, \mathrm{e}^{-ax} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for suitable $a > 0$ and $\nu > 0$.

3.2. Sum of product of Gaussians with zero average

Now take two independent random numbers $\hat{x}_1$ and $\hat{x}_2$ that are both distributed according to the distribution (2). Calculate (by evaluating the convolution integral) the probability density of the random variable

$$\hat{z} = \hat{x}_1^2 + \hat{x}_2^2 . \tag{4}$$

Again, write the result in the form (3). Can you guess the result when adding $r > 2$ independent random variables $\hat{x}_i^2$, all identically distributed.

Can you give the result a physical meaning? Hint: write $\hat{x}_i = \sqrt{m/2}\,\hat{v}_i$ where m is the mass of a gas particle and v_i the ith component of its velocity.