1. Problem: Ornstein-Uhlenbeck process (10 points)

The Wiener process describing the Brownian motion of a “free” particle, is non-stationary, i.e., does not allow for a meaningful time-independent $p_1(x)$. In order to permit a stationary solution, we have to “confine” the particle by adding a restoring force. The simplest way to do so is by means of a parabolic potential. In the stationary limit, i.e., for the initial preparation of the particle in the infinite past, this results in the so-called *Ornstein-Uhlenbeck process*, which we will consider now. It obeys a one-dimensional Fokker-Planck equation with constant diffusion (as the Wiener process) but additionally a linear drift term corresponding to the harmonic potential:

$$\frac{\partial}{\partial t}\,p(x,t) = \frac{\partial}{\partial x}\,[\kappa\,x\,p(x,t)] + D\frac{\partial^2}{\partial x^2}\,p(x,t)\,. \tag{1}$$

In order to obtain a stationary solution, we have to require that the otherwise arbitrary constants κ and D are positive. As usual, the conditional probability $p_{1|1}(x,t|x_0,t_0)$ is the solution of this equation with the initial condition $p(x,t_0) = \delta(x-x_0)$. For the one-time probability $p_1(x,t)$, we will require below, due to stationarity, $p_1(x,t) = p_{\mathrm{s}}(x)$.

As discussed in the lecture, a physical realization of this process (of course, in the approximative sense) is provided by the velocity fluctuations of a Brownian particle, which, in this context, is sometimes referred to as a Rayleigh particle. The linear term then corresponds to the linear damping of the particle, which is essentially proportional to its velocity (in the present notation x). In this case, κ and D are connected by a fluctuation-dissipation relation (Einstein relation): $D = \kappa k_B T/M$, where M is the mass of the particle.

1.1. Differential equation for the characteristic function

For the solution of the Fokker-Planck equation (1), we first introduce the characteristic function

$$G(k,t) = \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx}\,p(x,t)\,. \tag{2}$$

Proof that $G(k,t)$ fulfils the first-order partial differential equation

$$\frac{\partial}{\partial t}G(k,t) + \kappa\,k\,\frac{\partial}{\partial k}G(k,t) = -D\,k^2\,G(k,t)\,. \tag{3}$$

Which assumptions have to hold for $p(x,t)$ for $x \to \pm\infty$?

What is the initial condition for $G(k,t)$ corresponding to the conditional probability $p_{1|1}(x,t|x_0,t_0)$?

The solution of the linear, partial differential equation (3) is still non-trivial and will be discussed in the next problem. Here, we just give the result for $G(k,t)$ corresponding to $p_{1|1}(x,t|x_0,t_0)$:

$$G(k,t) = \exp\left\{-\frac{Dk^2}{2\kappa}\left[1 - e^{-2\kappa(t-t_0)}\right] + \mathrm{i}kx_0\,\mathrm{e}^{-\kappa(t-t_0)}\right\}. \tag{4}$$

1.2. Conditional probability and stationary solution

Derive from Eq. (4) the conditional probability $p_{1|1}(x,t|x_0,t_0)$ for the Ornstein-Uhlenbeck process?

Justity that $p_1(x,t) = p_\mathrm{s}(x) = \lim_{t_0\to-\infty} p_{1|1}(x,t|x_0,t_0)$ gives the correct one-time probability for the Ornstein-Uhlenbeck process, which is by definition a stationary process. Calculate $p_\mathrm{s}(x)$ from this relation.

Verify that the obtained $p_\mathrm{s}(x)$ is indeed a solution of the Fokker-Planck equation (1).

1.3. Autocorrelation function

Calculate the autocorrelation function $\langle\langle \hat{x}(t)\,\hat{x}(t')\rangle\rangle$ for the Ornstein-Uhlenbeck process.

Calculate and interpret the correlation time

$$\tau = \int_0^\infty \mathrm{d}t\,\frac{\langle\langle \hat{x}(t)\,\hat{x}(0)\rangle\rangle}{\langle\langle \hat{x}(0)^2\rangle\rangle}. \tag{5}$$

2. Problem: Method of characteristics (no points)

2.1. Solution of linear first-order partial differential equations

In the following, we consider a linear partial differential equation of first order, i.e., an equation of the form

$$a_1(\mathbf{x})\,\frac{\partial u}{\partial x_1} + a_2(\mathbf{x})\,\frac{\partial u}{\partial x_2} + \cdots + a_n(\mathbf{x})\,\frac{\partial u}{\partial x_n} = b(\mathbf{x})u\,. \tag{6}$$

Here, $u = u(\mathbf{x})$ is the unknown function which depends on n independent variables $\mathbf{x} = (x_1,\ldots,x_n)$ and the functions $\{a_i(\mathbf{x})\}$ and $b(\mathbf{x})$ are given. In applications in physics, typically one of the independent variables is the time and the others are space coordinates.

In order to specify the solution u, we have to supplant Eq. (6) by the value of u on an $n-1$ dimensional hypersurface Γ defined by the solution of an equation $g(\mathbf{x}) = 0$. The combination of this condition with the partial differential equation is called a Cauchy problem. Here, the vector field $\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), \ldots, a_n(\mathbf{x}))$ must never be parallel to the hypersurface, for reasons which should become clear in a moment.

At first sight surprinsingly, the solution of the Cauchy problem can be reduced to the solution of a system of ordinary differential equations. This becomes less surprising once one realizes that the equation (6) describes the flow of "particles" in the vector field $\mathbf{a}(\mathbf{x})$.[1] More concretely, consider the solutions $\mathbf{x}(s;\mathbf{x}_0)$ of the initial value problem

$$\frac{\mathrm{d}\mathbf{x}(s;\mathbf{x}_0)}{\mathrm{d}s} = \mathbf{a}(\mathbf{x}(s;\mathbf{x}_0)) \quad \text{with} \quad \mathbf{x}(s=0;\mathbf{x}_0) = \mathbf{x_0} \tag{7}$$

where $\mathbf{x}_0 \in \Gamma$ is arbitrary. This defines a set of so-called characteristic curves which "foliate" the n-dimensional space at least in the surrounding of the hypersurface Γ.

Then, the solution of the initial value problem along the characteristics

$$\frac{\mathrm{d}u(s;\mathbf{x}_0)}{\mathrm{d}s} = b(\mathbf{x}(s;\mathbf{x}_0))\, u(s;\mathbf{x}_0) \quad \text{with} \quad u(\mathbf{s}=0;\mathbf{x}_0) = u(\mathbf{x}_0) \tag{8}$$

in fact defines a function $u(\mathbf{x})$ in this region, namely by choosing the unique $\mathbf{x}_0$ and s with $\mathbf{x} = \mathbf{x}(s;\mathbf{x}_0)$ and then taking for $u(\mathbf{x})$ the value $u(s;\mathbf{x}_0)$. Note that this requires the solution of an implicit set of equations, which often is not readily possible.

Show that the so-defined $u(\mathbf{x})$ solves the partial differential equation (6).

Is it now clear, why the vector field $\mathbf{a}(\mathbf{x})$ must not be parallel to the surface Γ in any point?

Sketch a generic example for Γ, a vector field $\mathbf{a}(\mathbf{x})$ and a couple of characteristic curves in a two-dimensional space, i.e., for $n = 2$.

2.2. Application to Fokker-Planck equation of Ornstein-Uhlenbeck process

As an example, solve the Fourier transform (2) of the Fokker-Planck equation (1) of the Ornstein-Uhlenbeck process.

Enjoy!

[1] This method can be generalized to so-called quasilinear equations where the unknown u is still allowed to be present in the prefactors $\{a_i(\mathbf{x}, u)\}$ of the first derivatives and the right-hand side can assume the form $b(\mathbf{x}, u)$. Then the solution "flows" in an $n+1$ dimensional space.