1. Wiener process – revisited

The Wiener process $\hat{W}(t)$ describes the asymptotic behavior of the symmetric random walk for $t \to \infty$, $n \to \infty$ and n^2/t fixed. The evolution of the continuous position variables is then given by

$$p_{1|1}(W, t|W_0, t_0) = \frac{1}{\sqrt{2\pi(t - t_0)}} \exp\left[-\frac{(W - W_0)^2}{2(t - t_0)}\right], \tag{1}$$

$$p_1(W, 0) = \delta(W). \tag{2}$$

The process $\hat{W}(t)$ is Gaussian, time-homogeneous but not stationary. It describes the diffusion of a Brownian particle. Another important feature of the Wiener process are its independent increments.

1.1. One-time probability

What is the one-time probability $p_1(W, t)$ for the Wiener process defined above?

1.2. Conditional expectation values

Show that for $0 < t_1 \leq t_2$

$$\left\langle \hat{W}(t_2) \mid W_1, t_1 \right\rangle = W_1,$$
$$\left\langle\!\left\langle \left[\hat{W}(t_2)\right]^2 \mid W_1, t_1 \right\rangle\!\right\rangle = t_2 - t_1,$$

where the expectation value and the second cumulant are conditioned on the value $W_1 = \hat{W}(t_1)$ of the process at the earlier time t_1.

1.3. Correlations

Derive for $t_1, t_2 > 0$ the relations

$$\left\langle \hat{W}(t_1) \right\rangle = 0\,,$$
$$\left\langle \hat{W}(t_2)\hat{W}(t_1) \right\rangle = \min(t_1, t_2)\,,$$
$$\left\langle \left[\hat{W}(t_1) - \hat{W}(t_2)\right]\left[\hat{W}(t_3) - \hat{W}(t_4)\right] \right\rangle = \left|(t_1, t_2) \cap (t_3, t_4)\right|,$$

where the right-hand side of the third equation means the length of the overlap of both intervals. Interpret these results in connection with the property of independent increments that the Wiener process obeys.

2. Stochastic integrals

The stochastic integral $\int_{t_0}^{t} \hat{g}(t')\mathrm{d}\hat{W}(t')$ is defined as a kind of Riemann-Stieltjes integral. In contrast to the usual integral definition known from calculus, the infinite variation of the Wiener process implies that the value of the integral depends on the specific form of the discretization procedure. Here, we demonstrate this on the basis of the most simple example, namely the integral of the Wiener process itself.

Therefore we divide the interval $[t_0, t]$ into n subintervals $t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n-1} \leq t$ of equal length, i.e. $t_i = t_0 + i(t - t_0)/n$ and introduce a one-parametric family of stochastic integrals of the Wiener process defined by

$$\int\limits_{t_0}^{t} \hat{W}(t') \overset{\lambda}{\circ} \mathrm{d}\hat{W}(t') := \sum_{i=1}^{n} \left[\lambda \hat{W}(t_i) + (1-\lambda)\hat{W}(t_{i-1})\right]\left[\hat{W}(t_i) - \hat{W}(t_{i-1})\right]. \tag{3}$$

The case $\lambda = 0$ corresponds to the so-called Itô integral, while $\lambda = 1/2$ yields the Stratonovich definition of the stochastic integral.

2.1. Expectation value of the generalized stochastic integral

Calculate the mean value of the integrals

$$\left\langle \int\limits_{t_0}^{t} \hat{W}(t') \overset{\lambda}{\circ} \mathrm{d}\hat{W}(t') \right\rangle \tag{4}$$

What is the result for $\lambda = 0$, i.e., for the Itô integral. How could one have seen this beforehand?

2.2. Correlation formula for the Itô integral

In general the Itô integral is defined for an arbitrary non-anticipating function $\hat{g}(t)$ by

$$\int\limits_{t_0}^{t} \hat{g}(t')\,\mathrm{d}\hat{W}(t') := \sum_{i=1}^{n} \hat{g}(t_{i-1})\left[\hat{W}(t_i) - \hat{W}(t_{i-1})\right]$$

with the times t_i as given above. Calculate the mean value

$$\left\langle \int\limits_{t_0}^{t} \hat{g}(t')\,\mathrm{d}\hat{W}(t') \int\limits_{t_0}^{t} \hat{h}(t')\,\mathrm{d}\hat{W}(t') \right\rangle .$$

where $\hat{h}(t)$ is another, arbitrary, non-anticipating function.

Compare the above result to the mean value of

$$\left\langle \int\limits_{t_0}^{t} \hat{g}(t')\,\mathrm{d}\hat{W}(t') \left[\int\limits_{t_0}^{t'} \hat{h}(t'')\,\mathrm{d}\hat{W}(t'')\right] \right\rangle .$$

What is the difference between the two expressions?

3. Itô calculus

3.1. Itô Lemma

Calculate

$$\int_{t_0}^{t} \Big[\hat{W}(t') \Big]^n \, \mathrm{d}\hat{W}(t')$$

with the help of Itô's lemma:

$$\mathrm{d}f[\hat{W}(t),t] = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W^2} \right) dt + \frac{\partial f}{\partial W} \, d\hat{W}(t) \,.$$

What is the result of this stochastic integral in the special case $n = 1$?

Check the result for the mean value $\big\langle \int_{t_0}^{t} \hat{W}(t') \, \mathrm{d}\hat{W}(t') \big\rangle$.

Can you give an expression for $\big\langle \int_{t_0}^{t} [\hat{W}(t')]^n \, \mathrm{d}\hat{W}(t') \big\rangle$?

3.2. Example

Express the stochastic integral

$$\int_{t_0}^{t} \mathrm{e}^{\hat{W}(t')} \mathrm{d}\hat{W}(t')$$

in terms of a time-integral and a function of the Wiener process.