1. Dichotomic Noise or Random Telegraph Process

We want to consider a random process

$$\hat{x}(t) := a\,(-1)^{\hat{n}(t)}$$

i.e. a signal that has either value $+a$ or $-a$, where a is fixed, and $\hat{n}(t)$ refers to a Poisson process with jump rate λ.

1.1. Mean and Variance

What is the mean value $\langle \hat{x}(t) \rangle$ and the auto correlation $\langle \hat{x}(t)\hat{x}(t') \rangle$ of this process? What is the spectral density $S(\omega)$? In which limit does white noise emerge?

1.2. Probabilities

Due to the Markovian nature of the underlying Poisson process, the random telegraph process is also Markovian and, thus, fully described by the one-time probability and two-time conditional probability. Derive the conditional probability $p_{1|1}(x_2, t_2|x_1, t_1)$ $(t_1 < t_2)$ and the probability $p_1(x, t)$ for this process.

1.3. Master equation

How does the master equation of this process look like? Interpret and discuss the result.

2. Ornstein-Uhlenbeck Process

The Wiener process studied in the lecture is non-stationary, i.e. does not allow for a meaningful time-independent $p_1(x)$. Here, we now consider the simplest continuous, stationary Markov process, the so-called Ornstein-Uhlenbeck process. It obeys a one-dimensional Fokker-Planck equation with constant diffusion (as the Wiener process) but additionally a linear drift term, which acts as a restoring force:

$$\partial_t\, p_{1|1}(x, t|x_0, 0) = \partial_x \left[\kappa\, x\, p_{1|1}(x, t|x_0, 0)\right] + D\partial_x^2\, p_{1|1}(x, t|x_0, 0). \tag{1}$$

A physical realization of this process (of course, in the approximative sense) is provided by the velocity fluctuations of a Brownian particle, which, in this context, is sometimes referred to as Rayleigh particle. The linear term then corresponds to the linear damping of the particle, which is essentially proportional to its velocity (in the present notation x). Obviously, κ and D are then connected by a fluctuation-dissipation relation (Einstein relation): $D = \kappa k_B T/M$, where M is the mass of the particle.

2.1. Differential equation for the characteristic function

The Fokker-Planck equation for this process, can be most easily solved by introducing the characteristic function

$$G(k,t) = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{\mathrm{i}kx} \, p_{1|1}(x,t|x_0,0) \, .$$

Proof that $G(k,t)$ fulfils the first-order partial differential equation

$$\partial_t G(k,t) + \kappa \, k \, \partial_k G(k,t) = -D \, k^2 G(k,t). \tag{2}$$

Which assumptions have to hold for $p_{1|1}(x,t,|x_0,0)$ for $x \to \pm\infty$?

2.2. Solution

In order to solve a partial differential equation like Eq. (2) one can also solve the characteristic system (using the method of characteristics, explanation see section 3)

$$\frac{dt}{1} = \frac{dk}{\kappa k} = \frac{dG}{-Dk^2 G}.$$

Derive the characteristic function $G(k,t)$ from this equation system.

2.3. Mean and variance

Calculate the mean $\langle \hat{x}(t) \rangle$ and the variance $\mathrm{var}(\hat{x}(t)) = \langle (\hat{x}(t) - \langle \hat{x}(t) \rangle)^2 \rangle$ of this process with

$$G(k,t) = \exp\left[-\frac{Dk^2}{2\kappa}(1 - e^{-2\kappa t}) + \mathrm{i}kx_0 e^{-\kappa t} \right] .$$

What is the conditional probability $p_{1|1}(x,t|x_0,0)$ for the Ornstein-Uhlenbeck process?

2.4. Stationary solution

Derive the stationary solution $p_s(x)$ by setting $\partial_t p = 0$ in equation (1). Compare the result with the expression for the time-dependent solution for $t \to \infty$. Is the result as you expected it from the calculation of the mean and variance of the process?

Compare this damped process with the Boltzmann distribution for a free particle and derive the Einstein relation.

3. Method of characteristics

In order to solve a partial differential equation (PDE) of the form

$$\partial_t G(k,t) + \kappa\, k\, \partial_k G(k,t) = -D\, k^2 G(k,t)$$

we introduce a function $V(k,t,G)$ that fulfills the homogeneous PDE

$$\partial_t V(k,t,G) + \kappa\, k\, \partial_k V(k,t,G) - D\, k^2 \partial_G V(k,t,G) = 0$$

To solve the homogenous equation we assume that the coordinates of $V(s) = V(k(s), t(s), G(s))$ depend on the parameter s. Thus we can write

$$\frac{\mathrm{d}V}{\mathrm{d}s} = \frac{\mathrm{d}k}{\mathrm{d}s}\frac{\mathrm{d}V}{\mathrm{d}k} + \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}G}{\mathrm{d}s}\frac{\mathrm{d}V}{\mathrm{d}G}$$

and consequently $\mathrm{d}V/\mathrm{d}s = 0$ if

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}k}{\mathrm{d}s} = \kappa k, \qquad \frac{\mathrm{d}G}{\mathrm{d}s} = -Dk^2.$$

Therefore we reduced the PDE to a system of ordinary differential equations (ODEs). This system of ODEs or equivalently the characteristic system

$$\mathrm{d}s = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}k}{\kappa k} = \frac{\mathrm{d}G}{-Dk^2}$$

remains to be solved.

Geometrical interpretation: The value of the solution $V(s)$ is constant along the characteristic curves of the partial differential equation given parametrically by $(k(s), t(s), G(s))$.