1. First-Passage problems and Gambler's ruin

In the following, we consider a one-dimensional discrete process described by the birth-death forward equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p(n,t|m,0) = \gamma^+_{n-1}\,p(n-1,t|m,0) + \gamma^-_{n+1}\,p(n+1,t|m,0) - \left(\gamma^+_n + \gamma^-_n\right)\,p(n,t|m,0) \qquad (1)$$

with the conditional probability $p(n,t|m,0)$ for the process to be at time t at position n provided that initially, at time $t=0$, we started at position m. Obviously, this implies the initial condition

$$p(n,t=0|m,0) = \delta_{n,m}\,. \qquad (2)$$

Often, the dynamics (1) is restricted to a subset of all integers and thus boundary conditions also come into play. For instance, one may be interested in the probability that the process reaches another site R (with, say, $R > m$) for the first time and, in case that this happens, after what time. This first-passage time is different for different realizations of the process and therefore a random quantity. In the following, we will derive relations for its average, the so-called mean first-passage time.

A classical example of Eq. (1) with boundary conditions is the gambler's ruin problem. If a gambler starts with an initial captial of m and the game continues until his capital is reduced to zero. How probable is that event and how long does it take on average until the game is finished?

1.1. Absorbing boundary conditions

We assume that the site R is an absorbing boundary such that whenever a random walker hits R he is out (i.e. does not longer contribute to $p(n,t|m,0)$ if you imagine $p(n,t|m,0)$ as an ensemble of random walkers all starting at $t=0$ in m). Thus Eq. (1) is only valid for $n < R-1$ and we have the additional condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\,p(R-1,t|m,0) = \gamma^+_{R-2}\;\,p(R-2,t|m,0) - \left(\gamma^+_{R-1} + \gamma^-_{R-1}\right)p(R-1,t|m,0)\,. \qquad (3)$$

The probability $p(R,t|m,0)$ does not appear anymore.

Please note that since transitions back to $R-1$ starting from R are not possible, the conservation of probability cannot be fulfilled,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=-\infty}^{R-1} p(n,t|m,0) = -\gamma^+_{R-1}\;p(R-1,t|m,0)\,. \qquad (4)$$

This can be interpreted as the density of an ensemble of independent particles each conducting a random walk until it disappears at $n = R$. The number of remaining particles decreases. [If you do not like this fact, introduce an absorbing state which captures the lost particles.]

In other words, the probability of reaching R, having started at m, is given by

$$\pi_R(m) = \int_0^\infty \mathrm{d}t \, \left(- \sum_{n=-\infty}^{R-1} \frac{\mathrm{d}}{\mathrm{d}t} p(n,t|m,0) \right) \tag{5}$$

Furthermore, provided that the process reaches R with probability one, i.e., $\pi_R(m) = 1$, we find the mean first-passage time

$$\tau_R(m) = \int_0^\infty \mathrm{d}t \, t \left(- \sum_{n=-\infty}^{R-1} \frac{\mathrm{d}}{\mathrm{d}t} p(n,t|m,0) \right) = \sum_{n=-\infty}^{R-1} \int_0^\infty \mathrm{d}t \; p(n,t|m,0) \, . \tag{6}$$

Try to understand these relations and the assumptions put into their derivation.

1.2. Backward equation for splitting probabilities and mean first-passage time

Many formalisms have been developed for the calculation of quantities relating to first-passage problems. Here, we shall derive one based on the backward equation, which for time-homogeneous processes assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} p(n,t|m,0) = \sum_{m'} \Gamma^{\dagger}_{mm'} \, p(n,t|m',0) \tag{7}$$

Derive this from the backward equation given in the lecture and also write down the adjunct $\Gamma^{\dagger}_{mm'}$ of the master operator for the birth-death process (1).

Derive for the splitting probability the equation

$$\sum_{m'} \Gamma^{\dagger}_{mm'} \, \pi_R(m') = 0 \tag{8}$$

with boundary condition $\pi_R(R) = 1$. Interpret the boundary condition. Rewrite this equation in the form

$$\pi_R(m) = \frac{\gamma_m^+}{\gamma_m^+ + \gamma_m^-} \, \pi_R(m+1) + \frac{\gamma_m^-}{\gamma_m^+ + \gamma_m^-} \, \pi_R(m-1). \tag{9}$$

and try to understand this result.

Similarly, show that the mean first-passage time obeys

$$\sum_{m'} \Gamma^{\dagger}_{mm'} \, \tau_R(m') = -1 \tag{10}$$

with boundary condition $\tau_R(R) = 0$. Again, discuss the plausibility of the boundary condition.

1.3. Two absorbing boundaries

Now we ask for the probability $\pi_R(m)$ and $\pi_L(m)$ that a process starting at m reaches first either some $R > m$ or $L < m$, respectively. Thus, we introduce additionally to the equation at R, an absorbing boundary at L:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, p(L+1,t|m,0) = \gamma^-_{L+2}\, p(L+2,t|m,0) - \left(\gamma^+_{L+1} + \gamma^-_{L+1}\right) p(L+1,t|m,0)\,. \qquad (11)$$

The splitting probability $\pi_R(m)$ now still obeys the backward equation (9) with the same boundary condition at $m = R$ but additionally, it has to fulfill $\pi_R(L) = 0$. Why?

The concept of the mean first-passage time can be generalized to a mean exit-time $\tau(m)$ if we do not care about the fact where the process leaves the “allowed” range $L+1 < m < R-1$. Make yourself plausible that then (10) holds (replacing $\tau_R(m)$ by $\tau(m)$) with the boundary conditions $\tau(L) = \tau(R) = 0$.

Think of a physical setup with two absorbing boundaries or a similar problem of the gambler’s ruin which has two boundary conditions.

1.3.1. Example: Gambler’s ruin

Consider the splitting probability $\pi_0(m)$ for $m > 0$ and $\gamma^+_m = \gamma^+$ and $\gamma^-_m = \gamma^-$. Derive the result

$$\pi_0(m) = \frac{x^R - x^m}{x^R - 1} \qquad (12)$$

where $x = \gamma^-/\gamma^+$.

Hint: Consider the difference equation for $\Delta_m = \pi_0(m+1) - \pi_0(m)$. Use $\sum_{m=0}^{R-1} \Delta_m = -1$ (why?).

This gives the probability that a gambler goes bancrupt starting with initial capital m (i.e., ends up with zero capital $L = 0$) if he plays against another gambler with capital $R - m$ with asymmetrical chances for winning the game—assuming a continuous version of the game. Actually, the result for the more realistic discrete-time process is the same.

Discuss this in the limit $\gamma^- = \gamma^+$ and also for $R \to \infty$, i.e. the gambler playing against a bank.

Show that the mean duration of the game is given in the “symmetric” case $\gamma^+ = \gamma^- = 1$ by

$$\tau(m) = m(R-m)/2 \qquad (13)$$

Study some numerical examples for this first-passage time. Is the duration of the game longer or shorter than you expected it?