1. Point process

We want to study a special class of random variables the so-called "point processes" (or "random set of points"). They model, for instance, the random occurrence of certain events like the passage of charge carriers through a device as a dot on a time axis.

1.1. Definitions

The simplest point process has a sample space described by a non-negative integer $s = 0, 1, \dots$ and for each s a set of s real number $\{\tau_1, \dots, \tau_s\}$. In particular, arbitrary permutations of the τ_i are identified. The probability distribution on this sample space is then defined by a sequence of non-negative functions $Q_s(\tau_1, \dots, \tau_s)$ which have to be symmetric in all arguments, and have to obey and fulfill the normalization condition

$$Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int \mathrm{d}\tau_1 \dots \mathrm{d}\tau_s \, Q_s(\tau_1, \dots, \tau_s) = 1 \,. \tag{1}$$

Here and in the following, all integrals run over the whole real axis and we also assume that the $Q_s(\tau_1, \dots, \tau_s)$ do not contain delta-functions of the type $\delta(\tau_1 - \tau_2)$, i.e., the probability that two events occur at the same time is zero.

A random variable $\hat{A}$ defined on this sample space consists of a sequence of functions $A_s(\tau_1, \dots, \tau_s)$, which, again, have to be symmetric in all arguments. The corresponding average is given by

$$\langle \hat{A} \rangle = A_0 \, Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int \mathrm{d}\tau_1 \dots \mathrm{d}\tau_s \, A_s(\tau_1, \dots, \tau_s) \, Q_s(\tau_1, \dots, \tau_s) \,.$$

1.2. Example: Number of events in a given interval

Let $\hat{N}$ be the number of points in a given interval $[t_a, t_b]$. In order to define the corresponding sequence of functions $N_s(\tau_1, \dots, \tau_s)$, we introduce the indicator $\chi(t)$ of this interval by

$$\chi(t) = \begin{cases} 1 & t_a < t < t_b \\ 0 & \text{otherwise.} \end{cases}$$

Then, the random variable $\hat{N}$ is represented by

$$N_s(\tau_1, \dots, \tau_s) = \sum_{\sigma=1}^{s} \chi(\tau_\sigma) \,.$$

1.2.1. Calculation of averages

Calculate the average $\langle \hat{N} \rangle$ and the mean square $\langle \hat{N}^2 \rangle$ for this example.

1.3. The Poisson distribution

A point process is called *independent* when each Q_s factorizes, i.e., for $s \geq 1$

$$Q_s(\tau_1, \ldots, \tau_s) = \mathrm{e}^{-\nu}\, q(\tau_1) \cdots q(\tau_s) \tag{2}$$

and $Q_0 = \exp(-\nu)$. Here, $q(\tau)$ is an arbitrary non-negative integrable function and the normalization constant $\exp(-\nu)$ is determined by

$$\nu = \int\limits_{-\infty}^{\infty} \mathrm{d}\tau\, q(\tau)\,. \tag{3}$$

1.3.1. Normalization

Proof that for independent processes (2) the Q_s fulfill the normalization condition (1) with the constant ν given in Eq. (3).

1.3.2. Example

Calculate $\langle \hat{N} \rangle$ and $\langle \hat{N}^2 \rangle$ for the example in section 1.2 in the case of independent point processes.

1.3.3. Probability distribution

Derive for the characteristic function of $\hat{N}$ the relation

$$\langle \mathrm{e}^{ik\hat{N}} \rangle = \exp\left[(\mathrm{e}^{ik} - 1) \int\limits_{t_a}^{t_b} q(\tau)\, \mathrm{d}\tau \right]$$

and from there the probability distribution of $\hat{N}$. Argue why this point process is a Poisson process where the parameter a from problem set 1 is the mean $\langle \hat{N} \rangle$ given by

$$a = \langle \hat{N} \rangle = \int\limits_{t_a}^{t_b} q(\tau)\, \mathrm{d}\tau.$$

1.3.4. Shot noise as stationary and independent point process

A stationary and independent point process is called *shot noise*. Stationary means that the function $q(\tau)$ is independent of the time argument τ. Since the normalization condition (3) cannot be fulfilled in this case, stationarity can only be defined in terms of a limit procedure:

$$q(\tau) = \begin{cases} \rho & |\tau| < T \\ 0 & \text{otherwise,} \end{cases}$$

where the limit is taken such that $T \to \infty, \nu \to \infty$, $\rho = \text{const}$. Derive ρ as a function of ν and T.

2. Shot noise

2.1. Campbell's process

Imagine a process of the form

$$\hat{y}(t) := \sum_{\sigma=1}^{\hat{s}} u(t - \hat{\tau}_\sigma) \tag{4}$$

where $u(t)$ is a given integrable function, which represents the response to an event at time $t = 0$. If the random set $\{\hat{\tau}_\sigma\}$ with density ρ is independent and stationary, then this type of process is called "Campbell's process".

Show that in this case

$$\langle \hat{y}(t) \rangle = \rho \int_{-\infty}^{\infty} d\tau \; u(\tau),$$

$$\langle\langle \hat{y}(t)\hat{y}(t') \rangle\rangle = \rho \int_{-\infty}^{\infty} d\tau \; u(\tau)u(t - t' + \tau).$$

2.1.1. Fourier transformation

The Fourier transform of the function $u(t)$ and the autocorrelation function $S_{\hat{y}}(t - t') = \langle\langle \hat{y}(t)\hat{y}(t') \rangle\rangle$ are defined as

$$u(\omega) := \int \mathrm{d}\tau \, \mathrm{e}^{i\omega\tau} u(\tau),$$

$$S_{\hat{y}}(\omega) := \int \mathrm{d}\tau \, \mathrm{e}^{i\omega\tau} S_{\hat{y}}(\tau),$$

where $S_{\hat{y}}(\omega)$ is denoted the spectral density of $\hat{y}$.

Show that for the Campbell's process the following relation is valid

$$S_{\hat{y}}(\omega) = \rho |u(\omega)|^2.$$

2.2. Schottky's theorem

The time dependent random function is now assumed to be the fluctuating part of an electric current, $\hat{y}(t) \equiv I(t)$. Electrons are assumed to be transported at independent random times. How does the function $u(t)$ in equation (4) thus look like?

Derive the Schottky relation that the shot noise $S_I(\omega)$ obeys

$$S_I(\omega) = e\langle \hat{I} \rangle.$$

Thus the Fano factor $F = S_I/e\langle I \rangle$ is unity. Interpret this result.

2.3. MacDonald's theorem

The charge transported in a time t by e.g. an electric current is given by $\hat{N}_{\hat{y}}(t) = \int_0^t \mathrm{d}t' \hat{y}(t')$. Proof MacDonald's theorem

$$S_{\hat{y}}(\omega) = \omega \int_0^\infty \mathrm{d}t \, \sin(\omega t) \, \frac{d}{dt} \langle\langle \hat{N}_{\hat{y}}(t)^2 \rangle\rangle \, . \tag{5}$$

The upper limit of Eq. (5) is guaranteed by a converging factor $\mathrm{e}^{-\varepsilon t}$. What does this imply for the Schottky process described in section 2.2?

Hint: To proof Eq. (5) first derive the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle\langle \hat{N}_{\hat{y}}(t^2) \rangle\rangle = 2 \int_0^t \mathrm{d}\tau \; S_{\hat{y}}(\tau)$$