1. Geometric Brownian motion with time-dependent coefficient

The linear SDE

$$\mathrm{d}\hat{x} = \alpha(t)\,\hat{x}\,\mathrm{d}t + \beta(t)\,\hat{x}\,\mathrm{d}\hat{W}(t) \tag{1}$$

describes the so-called geometric Brownian motion with arbitrary time-dependent drift and diffusion coefficients. It is used e.g. as model for price fluctuations of stocks and the basis of the Black-Scholes model.

1.1. Solution for $\hat{x}(t)$

Write down the solution in dependence of the initial condition for the process $\hat{x}(t_0)$ at time t_0.

Hint: Consider the equation for $\hat{y}(t) := \ln \hat{x}(t)$.

1.2. Expectation value

Calculate mean $\langle \hat{x}(t) \rangle$ and second moment $\langle [\hat{x}(t)]^2 \rangle$ for the geometric Brownian motion with time-dependent coefficients. Conclude from this result how the higher moments $\langle [\hat{x}(t)]^n \rangle$ will look like.

1.3. Stratonovich SDE

Assume that the SDE is

$$\mathrm{d}\hat{x} = \alpha(t)\,\hat{x}\,\mathrm{d}t + \beta(t)\,\hat{x} \circ \mathrm{d}\hat{W}(t)$$

given in the Stratonovich calculus, where we set for the moment the drift term to zero. Solve again for $\hat{x}(t)$ and derive the expression for the mean $\langle \hat{x}(t) \rangle$ and the second moment $\langle [\hat{x}(t)]^2 \rangle$.

Compare the result with the one from the Itô interpretation.

1.4. Uniqueness of the solution

One can readily proof the uniqueness of the solution of the SDE (1) by showing that for two solutions $\hat{x}_1(t)$ and $\hat{x}_2(t)$ the differential of their ratio vanishes:

$$\mathrm{d}[\hat{x}_1(t)/\hat{x}_2(t)] = 0\,.$$

To verify this relation, derive first an SDE for $1/\hat{x}_2(t)$ using the Itô formula. Then use the product rule to obtain the desired result.

Show that this guarantees the uniqueness of the solution for all times t, if one specifies the process at an arbitrary time t_0.

1.5. Inhomogeneous process

One can even solve a more general equation containing “inhomogeneous” terms,

$$\mathrm{d}\hat{x} = \big[\alpha(t)\,\hat{x} + \gamma(t)\big]\,\mathrm{d}t + \big[\beta(t)\,\hat{x} + \delta(t)\big]\,\mathrm{d}\hat{W}(t)$$

by making the ansatz $\hat{z}(t) = \hat{x}(t)\,\hat{K}(t)^{-1}$, where $\hat{K}(t)$ is obtained by writing the solution $\hat{x}_\mathrm{h}(t)$ of the homogeneous equation (1) as $\hat{x}_\mathrm{h}(t) = \hat{K}(t)\,\hat{x}_\mathrm{h}(0)$.

2. Options on futures

Often, the underlying asset of an option is not the cash product itself but rather a futures contract, which is often more liquid and also requires lower transaction fees.

We assume that the price of the futures contract entered at time t is given in terms of the price S of the underlying asset by the corresponding forwards price relationship

$$K = S\,\mathrm{e}^{r(T-t)}\,, \tag{2}$$

where T is the delivery time of the asset.

The price f of an option on the future, now depends on the futures price K (and t), i.e., $f = f(K,t)$, and thus only indirectly on the asset price S.

Derive using Eq. (2) from the Black-Scholes equation (in terms of S and t) the following partial differential equation for $f(K,t)$:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\,\sigma^2\,K^2\,\frac{\partial^2 f}{\partial K^2} - rf = 0$$

Rederive this equation along the lines of the original derivation of the Black-Scholes equation.

Hints:

Derive first the SDE of the futures price from the one of the underlying.

Then, use Itô’s lemma to derive the SDE for the price of the option on the future.

Setup an appropriate risk-free portfolio.

What is the risk-free return for this portfolio?