1. Probability generating function, factorial moments and cumultants

1.1. The Poisson distribution

The Poisson distribution is defined as

$$p_n = \frac{a^n}{n!} \mathrm{e}^{-a} \tag{1}$$

on the discrete range $n = 0, 1, 2, ...$

1.1.1. Example

Think of an example which type of process a Poisson distribution could describe.

1.1.2. Cumulants

Calculate the cumulants κ_m of the Poisson distribution, Eq. (1), defined by

$$\log G(k) = \sum_{m=0}^{\infty} \frac{(\mathrm{i}k)^m}{m!} \kappa_m \,, \tag{2}$$

where $G(k) = \sum_{n=0}^{\infty} p_n \mathrm{e}^{\mathrm{i}kn}$ is the characteristic function for a discrete random variable.
Hint: Use therefore the derivative of $\log G(k)$.

1.2. Probability generating function

The *probability generating function* of a random variable $\hat{x}$ is defined by

$$F(z) = \left\langle z^{\hat{x}} \right\rangle \tag{3}$$

where z is a complex number on the unit circle $|z| = 1$.
What condition does $\hat{x}$ have to fulfil in order that $F(z)$ is even defined for all z within the unit circle? What could be the advantage of using $F(z)$ compared to the characteristic function $G(k) = \langle \mathrm{e}^{ik\hat{x}} \rangle$?

1.2.1. Factorial moments

We define the *factorial moments* Φ_m by

$$\begin{aligned} \Phi_0 &= 1 \qquad (4) \\ \Phi_m &= \langle \hat{x}(\hat{x}-1)(\hat{x}-2)\dots(\hat{x}-m+1)\rangle \quad (m \geq 1) \qquad (5) \end{aligned}$$

Show that the factorial moments are also generated by

$$F(1-y) = \sum_{m=0}^{\infty} \frac{(-y)^m}{m!}\Phi_m. \qquad (6)$$

1.2.2. Factorial cumulants

The *factorial cumulants* θ_m are defined by

$$\log F(1-y) = \sum_{m=1}^{\infty} \frac{(-y)^m}{m!}\theta_m. \qquad (7)$$

What is the relation between the factorial cumulants and the factorial moments? Derive the expression up to κ_3.

1.2.3. Poisson distribution

What is the probability generating function $F(1-y)$ for the Poisson distribution, Eq. (1)?
How do the factorial moments look like?
What is special about the factorial cumulants of the Poisson distribution?

2. Gamma distribution

The family of gamma distributions is defined by

$$P(x) = \frac{a^\nu}{\Gamma(\nu)} x^{\nu-1} \mathrm{e}^{-ax} \qquad (8)$$

for $a > 0$, $\nu > 0$, $0 < x < \infty$.

2.1. Transformation of variables

Let the variables $\hat{x}_1, \hat{x}_2, \ldots \hat{x}_r$ be Gaussian with zero average and variance σ^2, and independent. Prove that

$$\hat{y} = \hat{x}_1^2 + \hat{x}_2^2 + \ldots + \hat{x}_r^2 \tag{9}$$

is gamma distributed.
Hint: Check first if $\hat{x}_i^2$ is gamma distributed. What is the characteristic function of the sum of two variables?

2.2. Example

Think of an example of the transformation $\hat{y} = \hat{x}_1^2 + \ldots \hat{x}_r^2$ for $r = 3$.
Hint: Statistical mechanics