Landauer formula

Motivation In conductors with a linear dimension comparable with the coherence length, the transport is governed by quantum effects. The transport then proceeds by (coherently) tunneling between the source and drain electrode via the states in the sample. The Landauer formula describes the conduction in terms of tunneling.

Exercise 1: Adiabatic transport model (4 points) Consider the geometry of a two-dimensional quantum point contact, i.e., a constriction in a two-dimensional electron gas. The Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m^*}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+V(x,y)\right]\psi(x,y)=E\psi(x,y), \tag{1}$$

where the potential $V(x,y)$ describes the lateral potential producing the constriction in the transverse $y-$ direction. The adiabatic approximation accounts for the fact that the spatial variation in the x direction is much slower than in the transverse y direction. In order to use this approximation, factor the wave function $\psi(x,y)$ into an $x-$ dependent part $\phi_n(x)$, and an $x-$ and $y-$ dependent transverse part $\chi_n(x,y)$ that is a solution of a one-dimensional Schrödinger equation at position x along the constriction. Hint: this is a simple separation of variables, substitute $\psi(x,y)=\phi(x)\chi(x,y)$ in the equation (1) and assume that the derivatives with respect to x are much smaller than the derivatives with respect to y. At any point x, the set of all possible transverse functions $\chi_n(x,y)$ forms a complete set, so that you can use it as a basis

$$\psi(x,y)=\sum_n \phi_n(x)\chi_n(x,y). \tag{2}$$

Multiply the resulting equation by $\chi_m^*(x,y)$, and integrate over all y. Argue that the resulting equation can be approximated by

$$\left[-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial x^2}+E_n(x)\right]\phi_n(x)=E\phi_n(x). \tag{3}$$

Interpret this result. Does it mean that the transport can be considered as a tunneling through a set of one-dimensional barriers $E_n(x)$, where each barrier comes from a confined level in the transverse direction?

Exercise 2: Single channel conductance (3 points) The transport through a one-dimensional channel proceeds through tunneling. The probability of transmission is given by $T(E)$. Argue that the current is then (assume spinless electrons)

$$I=\frac{e}{2\pi}\left[\int_0^\infty \mathrm{d}k v(k) f_1(k) T(E) - \int_0^\infty \mathrm{d}k' v(k') f_2(k') T(E')\right], \tag{4}$$

where the constant is the one-dimensional density of states in the $k-$space, $v(k)$ is the velocity, $T(E)$ is the transmission coefficient, and f_1 and f_2 are the reservoir distribution functions. Show that at low temperatures, the current between the leads at chemical potentials μ_1 and μ_2 is

$$I=\frac{e}{2\pi\hbar}\int_{\mu_1}^{\mu_2}\mathrm{d}E T(E). \tag{5}$$

Exercise 3: Interference of paths (3 points) When the tunneling can proceed through a pair of paths with transmission amplitudes t_1, t_2, the incoherent transmission coefficient is simply $T = T_1 + T_2 = |t_1|^2 + |t_2|^2$. In general, however, the tunneling amplitudes can interfere and the transmission is $T = |t_1 + t_2|^2$. The complex tunneling amplitudes have the phases ϕ_1, ϕ_2, i.e., $t_1 = |t_1|e^{i\phi_1}$, and $t_2 = |t_2|e^{i\phi_2}$. Find the dependence of the transmission coefficient T on the relative phase $(\phi_1 - \phi_2)$. In the Aharonov-Bohm effect, the relative phase between the two tunneling paths is given by the magnetic flux Φ encircled by the paths. Remember that the magnetic flux introduces the phase difference of $(e/\hbar)\Phi$, and show that the current through a ring is a periodic function of the flux.

Exercise 4: Connection to Meir-Wingreen formula, linear response (2 points) In the lectures you have seen the general formula for the current through a central region

$$I = \int d\omega \frac{1}{2\pi} \big(f_1(\omega) - f_2(\omega)\big) \sum_n \Gamma_n(\omega) A_n(\omega), \tag{6}$$

where n labels the states of the central region, $\Gamma_n = \Gamma_{1n}\Gamma_{2n}/(\Gamma_{1n} + \Gamma_{2n})$ is the effective tunneling rate between reservoirs 1 and 2, and A_n is the spectral function in the central region (we assume everything diagonal in n here).

Assume that the dependence on ω of Γ and A is weak and show that at low temperatures ($k_B T \ll eV$ with V the voltage drop between the reservoirs) $I = VG$ with G the conductance.

Explain then how (6) relates to (5).