Polarization bubble in 3 dimensions

Motivation The **polarization bubble** is one of the most important pieces appearing in Feynman diagrams, and is the essential constituent of the so-called Random Phase Approximation (RPA).

Exercise 1: Polarization bubble (10 points) Assume that an unperturbed electron system is described by the Hamiltonian

$$H_0 = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \, c^\dagger_{\mathbf{k}} c_{\mathbf{k}}, \tag{1}$$

with $\mathbf{k}$ the momentum and $k = |\mathbf{k}|$. For simplicity we assume that the electrons are spinless here.

(a) Consider the density-density correlator,

$$\chi_0(\mathbf{x}, t; \mathbf{x}', 0) = \left\langle T\{\psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t)\psi^\dagger(\mathbf{x}', 0)\psi(\mathbf{x}', 0)\}\right\rangle_0 . \tag{2}$$

Show that we can write, after a Fourier transformation $(\mathbf{x}, t) \to (\mathbf{q}, \omega)$

$$\chi_0(\mathbf{q}, \omega) = (-1)\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\int \frac{\mathrm{d}\omega'}{2\pi} iG^c_0(\mathbf{k}, \omega') \; iG^c_0(\mathbf{k}+\mathbf{q}, \omega+\omega') \equiv i\Pi^c_0(\mathbf{q}, \omega), \tag{3}$$

where we specifically choose 3 spatial dimensions. In particular justify the factor (-1). The function Π^c_0 is known as the "polarization function" or "polarization bubble".

(b) Draw a Feynman diagram corresponding to Π^c_0 and write the according labels to every propagator line and vertex. Observe that a "Kirchhoff law" corresponding to conservation of momentum and frequency holds at every node (vertex). It is recommended to make arrows on all lines to avoid confusion with the direction of the momentum and frequency "flow".

(c) Explicitly calculate now the ω' integral. [Set $\hbar = 1$ to make life easier.] Follow here the following lines (and answer the questions):

1. It is always recommended to do the frequency integrals first. Why?
2. The free Green's function is given by $G^c_0(\mathbf{k}, \omega) = \frac{\mathrm{e}^{i\omega\eta'}}{\omega - \epsilon_{\mathbf{k}} + i\eta_{\mathbf{k}}}$ with $\eta, \eta' > 0$ infinitesimal quantities and $\eta_{\mathbf{k}} = \eta \;\mathrm{sign}(\epsilon_{\mathbf{k}} - \epsilon_F)$. Why and when is the factor $\mathrm{e}^{i\omega\eta'}$ important?
3. The product $G^c_0(\mathbf{k}, \omega')G^c_0(\mathbf{k}+\mathbf{q}, \omega+\omega')$ is of the form $\frac{1}{\omega-a}\frac{1}{\omega-b}$. To avoid troubles with the case $a \to b$, it is recommended to write it in the form $\frac{1}{a-b}\left[\frac{1}{\omega-a} - \frac{1}{\omega-b}\right]$. Perform then the ω' integral by contour integration. Be extremely careful with the small shifts $\eta_{\mathbf{k}}$ and $\eta_{\mathbf{k}+\mathbf{q}}$, which also appear in the factor $\frac{1}{a-b}$.
4. Observe that the result is very similar to the Lindhard function of Series 5, Exercise 2. What precisely is the difference?

5. Split the result into real and imaginary parts using the formula $\frac{1}{\omega \pm i\eta} = \mathcal{P}\frac{1}{\omega} \mp i\pi\delta(\omega)$.

(d) Proceed with the calculation of the $\mathbf{k}$ integral in $\mathrm{Re}\Pi_0^c$. For this we make the assumption that the dispersion is given by $\epsilon_{\mathbf{k}} = \frac{k^2}{2m^*}$ with m^* the effective mass of the Fermi liquid described by H_0. Fermi momentum and Fermi energy then relate as follows: $\epsilon_F = \frac{k_F^2}{2m^*}$, and the density of states at the Fermi surface is $\nu(\epsilon_F) = \frac{1}{2\pi^2}(2m^*)^{3/2}\epsilon_F^{1/2}$.
Use then that $\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} = \frac{1}{2m^*}(q^2 + 2kq\cos\theta)$, with θ the angle between $\mathbf{k}$ and $\mathbf{q}$. If you set $x_0 = \frac{\omega}{4\epsilon_F}$, $x = \frac{q}{2k_F}$, $y = \frac{k}{2k_F}$, and $\kappa = \cos\theta$, then you can write (with g here an arbitrary function) $\int \mathrm{d}\mathbf{k}\, g(k, \cos\theta) = \int_0^\infty \mathrm{d}k k^2 \int_{-1}^1 \mathrm{d}(\cos\theta)\, g(k,\cos\theta) = (2k_F)^3 \int_0^\infty \mathrm{d}y y^2 \int_{-1}^1 \mathrm{d}\kappa\, g(y,\kappa)$.
You should then obtain

$$\mathrm{Re}\Pi_0^c(\mathbf{q},\omega) = \frac{k_F^3}{2\pi^2\epsilon_F}\mathcal{P}\int_0^{1/2} \mathrm{d}y y^2 \int_{-1}^1 \mathrm{d}\kappa \left[\frac{1}{x_0 - (x^2 + 2xy\kappa)} - \frac{1}{x_0 + (x^2 - 2xy\kappa)}\right]. \quad (4)$$

The final integration can be done using

$$\int \mathrm{d}x \frac{1}{ax+b} = \frac{1}{a}\ln(ax+b), \qquad \int \mathrm{d}x \ln(ax+b) = \frac{1}{a}\left[(ax+b)\ln(ax+b) - ax\right], \quad (5)$$

and

$$\int \mathrm{d}x x \ln(ax+b) = \frac{bx}{2a} - \frac{x^2}{4} + \left(\frac{x^2}{2} - \frac{b^2}{2a^2}\right)\ln|ax+b|. \quad (6)$$

You should then obtain the result

$$\mathrm{Re}\Pi_0^c(\mathbf{q},\omega) = -\nu(\epsilon_F)\left[\frac{1}{2} + \frac{f(x,x_0) + f(x,-x_0)}{8x}\right] \quad (7)$$

with

$$f(x,x_0) = \left[1 - \left(\frac{x_0}{x} - x\right)^2\right]\ln\left|\frac{x + x^2 - x_0}{x - x^2 + x_0}\right|. \quad (8)$$

(e) Now turn to $\mathrm{Im}\Pi_0^c$. Do the $\mathbf{k}$ integration by noting that you can write

$$\mathrm{Im}\Pi_0^c(\mathbf{q},\omega) = \pi\left|\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} n(\epsilon_k)\left[\delta(\omega + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}) - \delta(\omega + \epsilon_{\mathbf{k}-\mathbf{q}} - \epsilon_{\mathbf{k}})\right]\right|, \quad (9)$$

with $n(\epsilon_{\mathbf{k}}) = \theta(\epsilon_F - \epsilon_{\mathbf{k}})$ the Fermi function at zero temperature. Use again $\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} = \frac{1}{2m^*}(q^2 + 2kq\cos\theta)$ and the dimensionless variables x, x_0, y and κ. If you note furthermore that $\delta\left(g(x - x_0)\right) = \frac{1}{|g'(x_0)|}\delta(x - x_0)$ if $g(x)$ has a unique zero at $x = x_0$, then you should obtain

$$\mathrm{Im}\Pi_0^c(\mathbf{q},\omega) = \frac{k_F^3}{2\pi\epsilon_F}\left|\int_0^{1/2} \mathrm{d}y y^2 \int_{-1}^1 \mathrm{d}\kappa \left[\delta(x_0 - x^2 - 2xy\kappa) - \delta(x_0 + x^2 - 2xy\kappa)\right]\right|, \quad (10)$$

which eventually leads to (for $\omega \geq 0$)

$$\mathrm{Im}\Pi_0^c(\mathbf{q},\omega) = \begin{cases} \nu(\epsilon_F)\left|\frac{\pi}{8x}\left[1 - \left(\frac{x_0}{x} - x\right)^2\right]\right| & \text{for } |x - x^2| < x_0 < x + x^2, \\ \nu(\epsilon_F)\frac{\pi}{2}\left|\frac{x_0}{x}\right| & \text{for } 0 < x_0 < x - x^2, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$