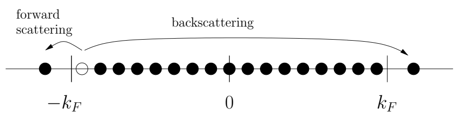
Theoretische Festkörperphysik – Herbstsemester 2010 **Exercise series 1**

Assistant: K. van Hoogdalem 01.10.2010

Linear Hydrodynamics and Bosonization

Motivation We present a very useful tool of many-body physics called bosonization in the guise of hydrodynamics. Starting from the classical equations of motion of a liquid, we arrive at the quantum description of density fluctuations in a fluid. This approach shows that, basically with the knowledge you already have, you can approach a novel technique that is relevant in contemporary research. Please notice the approach employed in this example: The description of a system of many particles is reduced to only few elementary excitations at low energies.



The physics of 1D fermionic systems (quantum wires, nanotubes, 1D optical lattices of ultracold fermions) is very different from the higher dimensional ones. The difference is mainly due to the reduced phase space for particle scattering: After collision of two particles, the Pauli principle forbids to scatter to an occupied state, and empty states are rare because of the dimension (see the figure on the right): A change of momentum k cannot be achieved without a change of energy ϵ_k. As the figure shows, there are essentially two remaining scattering processes: forward scattering with small momentum transfer, and backscattering with momentum transfer close to $\pm 2k_F$. Backscattering is an exchange interaction process, and it turns out that it is usually irrelevant, which means that it is absent in a low energy description of the system. The remaining (classical) forward scattering processes can be seen as density excitations of a particle-hole pair, which at first approximation has a bosonic nature (one also speaks of "polarons"). A very powerful way of describing the low energy physics (i.e. when the excitations remain all close to $\pm k_F$) is to treat those excitations as true bosons. This is known as *bosonization*.

Exercise 1: Linear Hydrodynamics (3 points) Consider a 1D system of length L with periodic boundary conditions, containing N spinless fermions. The particle (number) density is $\rho(x,t) = \rho_0 + \delta\rho(x,t)$, where $\rho_0 = N/L$ is the average density and $\delta\rho(x,t)$ the density fluctuations. Classically, small density fluctuations around the equilibrium state can be described by a linearized Navier-Stokes equation (see e.g. L. Reichl, *A Modern Course in Statistical Physics*, J. Wiley & Sons, 1998; Chap. 10):

$$\rho_0 \partial_t \big(mv(x,t)\big) = -\partial_x \big(\delta p(x,t)\big), \tag{1}$$

with m the mass of the particles, $v(x,t)$ the velocity field, and $\delta p(x,t)$ the pressure fluctuations in the system. Particle conservation is expressed through the continuity equation

$$\partial_t \rho(x,t) + \partial_x j(x,t) = 0 \tag{2}$$

with $j(x,t) = \rho_0 v(x,t)$ the current density.

(a) Justify the expression $\delta p = \delta\rho(x,t)/\rho_0\kappa_T$, where $\kappa_T = [\rho_0(\frac{\partial p}{\partial \rho})_{T,\text{equil.}}]^{-1}$ (i.e. which are the assumptions?).

(b) Show that Eqs. (1) and (2) can be combined to the wave equation

$$\partial_t^2\big(\delta\rho(x,t)\big) - c^2\partial_x^2\big(\delta\rho(x,t)\big) = 0 \tag{3}$$

and calculate its Fourier transform for the density fluctuation modes $\rho_k(t) = \int \mathrm{d}x\, \mathrm{e}^{-ikx}\delta\rho(x,t)$. What is the interpretation of $c^2 = 1/(m\rho_0\kappa_T)$?

Exercise 2: Lagrange and Hamilton Functions (4 points)

(a) In order to quantize, we want to identify the Fourier transformed wave equation with the Euler-Lagrange equation of a Lagrange function

$$\mathcal{L} = \frac{1}{2L}\sum_k \Big[m\big(\partial_t\phi_k(t)\big)^2 - m\omega_k^2\phi_k^2(t)\Big] \tag{4}$$

Take as an imput here that the dimension of ϕ_k is $(\text{Length})^{3/2}$ (to be worked out in more detail in the next exercise series). Determine the relation between ρ_k and ϕ_k and the form of ω_k. How can we interpret the spectrum ω_k?

(b) Calculate the Hamilton function H by introducing the conjugate momenta $\pi_k = \frac{\delta \mathcal{L}}{\delta(\partial_t \phi_k)}$ and performing the Legendre transformation $H = \sum_k \pi_k \, \partial_t \phi_k - \mathcal{L}$. How can we interpret the result?

Exercise 3: Quantization and Ground State (3 points) Quantization is imposed by replacing ϕ_k and π_k by the quantum fields $\hat{\phi}_k$ and $\hat{\pi}_k$ that satisfy the canonical commutation relations $[\hat{\phi}_k, \hat{\pi}_{k'}] = i\hbar L \delta_{k,k'}$ and $[\hat{\phi}_k, \hat{\phi}_{k'}] = [\hat{\pi}_k, \hat{\pi}_{k'}] = 0$. Show that the ground state wave function is described by

$$\Psi(\{\phi_k\}) = \langle \{\phi_k\} | \Psi \rangle = \prod_{k>0} \left[\frac{m\omega_k}{\pi L \hbar} \right]^{\frac{1}{4}} \mathrm{e}^{-m\omega_k |\phi_k|^2 / 2L\hbar}, \tag{5}$$

where ϕ_k is the eigenvalue of $\hat{\phi}_k$.

Exercise 4: Two-Body Interactions (2 points) Let us add the classical two-body interaction potential

$$V = \int \mathrm{d}x \mathrm{d}x' V(x - x') \rho(x) \rho(x') \tag{6}$$

and quantize it in the same way as before. What is its effect on the frequencies ω_k?

Exercise 5: One-Body Potential and Orthogonality Catastrophe (3 points) In the same way, we can add a one-body scattering potential

$$U = \int \mathrm{d}x U(x) \rho(x). \tag{7}$$

(a) Show that the Hamiltonian remains unchanged (up to a shift in energy) if we complete the square, i.e. change the boson modes by $\phi_k \to \phi_k + \Delta\phi_k$ with $\Delta\phi_k = 2U_{-k} L |k| / m\omega_k^2$. The new ground state function is thus $\Phi(\{\phi_k\}) = \Psi(\{\phi_k + \Delta\phi_k\})$.

(b) Show that the overlap between the old and new ground states is

$$\langle \Phi | \Psi \rangle = \exp\left(-\sum_{k>0} \frac{|U_k|^2}{\hbar m c^3 k} \right). \tag{8}$$

Observe that $k = n\pi/L$, and that the sum in the exponent diverges in the thermodynamic limit $L \to \infty$ (while keeping $\rho_0 = N/L$ constant). For a large but finite number of particles N, we have thus $\langle \Phi | \Psi \rangle \sim N^{-\alpha}$ with some exponent $\alpha > 0$ that depends directly on the potential U. The overlap is effectively zero, which is known as the "Orthogonality Catastrophe", and has been first investigated by P. W. Anderson in 1967 [Phys. Rev. Lett. **18**, 1049 (1967)]. It is a "catastrophe" because if $U(x)$ is abruptly (nonadiabatically) switched on at some time, there is no hope of finding the new ground state Φ by doing perturbation theory around the original ground state Ψ, because perturbation theory always produces a finite overlap between initial and finite states. This observation has stimulated a huge progress in the development of nonperturbative techniques, and most of the modern techniques we know (among them bosonization) have been applied and tested on the orthogonality catastrophe.