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Theoretical Solid-State Physics, Herbstsemester 2012

Blatt 1

Abgabe: 27. September, 12:00H (Treppenhaus 4. Stock)
Tutor: Patrick Hofer Zi.: 4.13

Die **6 Kreditpunkte** für Vorlesung und Übung erhält, wer 50% der Punkte aus den Hausaufgaben erreicht.

(1) **Some useful operator identities** (5 Punkte)

(a) Show that the commutation relation

$$[a_m^\dagger a_n, a_k^\dagger a_l] = \delta_{nk} a_m^\dagger a_l - \delta_{ml} a_k^\dagger a_n$$

holds for both bosonic and fermionic operators $a^\dagger$, a, with m, n, k, l being arbitrary single-particle quantum numbers, and δ_{nk} is the Kronecker delta.

(b) Show that for an arbitrary analytic operator function $f(A)$ it holds that

$$e^S f(A) e^{-S} = f(e^S A\, e^{-S}) \,,$$

a result that is easily generalized to the case where instead of A we have several operators.

Hint: As a first step, consider the case where $f(A) = A^n$ with integer number n.

(c) As an application of the general result in part b), show that for the Glauber displacement operator $D(\beta) \equiv \exp(\beta a^\dagger - \beta^* a)$ ($\beta \in \mathcal{C}$), generating coherent states when acting on the vacuum $|0\rangle$ of the bosonic mode a [reminder: $a|\beta\rangle = \beta|\beta\rangle$, $|\beta\rangle \equiv D(\beta)|0\rangle$], it holds that

$$D^\dagger(\beta) f(a, a^\dagger) D(\beta) = f(a+\beta, a^\dagger + \beta^*) \,,$$

where $f(a, a^\dagger)$ is again an arbitrary analytic function.

(2) **Bogoliubov transformation** (5 Punkte)

The effective Hamiltonian

$$H = \epsilon_c c^\dagger c + \epsilon_d d^\dagger d - \Delta dc - \Delta^* c^\dagger d^\dagger \,,$$

contains fermions in two kinds of states c und d (i.e., $\{c, c^\dagger\} = 1$, $\{c, d\} = 0$ etc., here, $\{,\}$ is the anticommutator). We would like to diagonalize this Hamiltonian, i.e., express it in the form

$$H = E_\gamma \gamma^\dagger \gamma + E_\delta \delta^\dagger \delta + E_0$$

by introducing the so-called quasiparticle operators α, β through the following unitary transformation:

$$c^\dagger = u^* \gamma^\dagger + v\delta\,, \quad d = -v^* \gamma^\dagger + u\delta\,.$$

(u, v are complex numbers, γ, δ are fermionic operators, i.e., obey $\{\gamma, \gamma^\dagger\} = 1$ etc. !)

(a) Show that the coefficients have to fulfill $|u|^2 + |v|^2 = 1$.

(b) Express H through γ and δ, and determine u and v such that H is diagonalized. Hint: introduce new variables ϕ, η by setting $u = \cos\eta$, $v = e^{i\phi}\sin\eta$.
Determine ϕ, η such that the "unwanted" terms like $\gamma^\dagger\delta^\dagger$ vanish. Find the energy spectrum of the new quasiparticles, that is, find the expressions for E_γ and E_δ in the special case $\epsilon_c = \epsilon_d = \epsilon$, and u, v, Δ are real.

(c) Discuss the meaning of E_γ, E_δ, and E_0.