Excitations in an interacting Fermi gas

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In the analysis of excitations in a non-interacting electron gas we used the imaginary part of the Lindhard polarization function. From the condition that $\mathrm{Im}\,\chi_0^R(\vec{q},\omega) \neq 0$ we found the particle-hole continuum., i.e., the part of the ω-$\vec{q}$ plane that comprises all possible particle-hole excitations. In the interacting case, we need to analyze the imaginary part of the polarization function

$$\chi^{RPA}(\vec{q},\omega) = \frac{\chi_0^R(\vec{q},\omega)}{1 - V_{\vec{q}}\,\chi_0^R(\vec{q},\omega)}$$

evaluated within the random-phase approximation (RPA). It is not hard to see that $\mathrm{Im}\,\chi^{RPA}(\vec{q},\omega)$ depends on both $\mathrm{Re}\,\chi_0^R(\vec{q},\omega)$ and $\mathrm{Im}\,\chi_0^R(\vec{q},\omega)$. Thus we will start with an explicit derivation of the expression for $\mathrm{Re}\,\chi_0^R(\vec{q},\omega)$.

The Lindhard polarization function

$$\chi_0^R(\vec{q},\omega) = \frac{1}{V}\sum_{\vec{k}\sigma} \frac{n_F(\xi_{\vec{k}}) - n_F(\xi_{\vec{k}+\vec{q}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} + \omega + i\eta} \quad (\eta \to 0^+),$$

i.e., since nothing under the last sum depends on the spin index,

$$\chi_0^R(\vec{q},\omega) = \frac{2}{V}\sum_{\vec{k}} \frac{n_F(\xi_{\vec{k}}) - n_F(\xi_{\vec{k}+\vec{q}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} + \omega + i\eta} \quad (\eta \to 0^+)$$

can be recast as

$$\chi_0^R(\vec{q},\omega) = \frac{2}{V}\sum_{\vec{k}} \frac{n_F(\xi_{\vec{k}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} + \omega + i\eta} \qquad (\eta \to 0^+)$$

$$- \frac{2}{V}\sum_{\vec{k}'} \frac{n_F(\xi_{-\vec{k}'})}{\xi_{-\vec{k}'-\vec{q}} - \xi_{-\vec{k}'} + \omega + i\eta}$$

in the second term we introduced new variable $\vec{k}' \equiv -\vec{k}-\vec{q} \implies \vec{k} = -\vec{k}'-\vec{q}$. We further notice that $\xi_{-\vec{k}} = \xi_{\vec{k}}$, and therefore $n_F(\xi_{-\vec{k}'}) = n_F(\xi_{\vec{k}'})$

$$\Rightarrow \chi_0^R(\vec{q},\omega) = \frac{2}{V}\sum_{\vec{k}} \frac{n_F(\xi_{\vec{k}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} + \omega + i\eta} \qquad (\eta \to 0^+)$$

$$- \frac{2}{V}\sum_{\vec{k}} \frac{n_F(\xi_{\vec{k}})}{\xi_{\vec{k}+\vec{q}} - \xi_{\vec{k}} + \omega + i\eta}$$

where we returned to the variable $\vec{k}$ in the second term. Thus

$$\chi_0^R(\vec{q},\omega) = \frac{2}{V}\sum_{\vec{k}} n_F(\xi_{\vec{k}})\left[\frac{1}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} + \omega + i\eta} + \frac{1}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} - \omega - i\eta}\right].$$

$(\eta \to 0^+)$

In particulare, at $T = 0$, where $n_F(\xi_{\vec{k}}) = \theta(k_F - |\vec{k}|)$, we have that $\chi_0^R(\vec{q},\omega) = S_1 + S_2$, where

$$S_1 \equiv \frac{2}{V}\sum_{\vec{k}} \frac{\theta(k_F - |\vec{k}|)}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} + \omega + i\eta}; \quad S_2 \equiv \frac{2}{V}\sum_{\vec{k}} \frac{\theta(k_F - |\vec{k}|)}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}} - \omega - i\eta}$$

$(\eta \to 0^+)$.

$$S_1 = \frac{(-2)}{(2\pi)^3} \int d^3\vec{k}\, \frac{\theta(k_F - |\vec{k}|)}{\frac{|\vec{q}|^2}{2m} + \frac{\vec{k}\cdot\vec{q}}{m} - \omega - i\eta}$$

$\Rightarrow$

$$S_1 = \frac{(-2) * 2m}{(2\pi)^3} \int d^3\vec{k}\, \frac{\theta(k_F - |\vec{k}|)}{2\vec{k}\cdot\vec{q} + |\vec{q}|^2 - \tilde{\omega}},$$

where $\tilde{\omega} \equiv 2m(\omega + i\eta) \equiv 2m\omega + i * \underbrace{2m\eta}_{\eta'\ (\eta' \to 0^+)}$

$\Rightarrow$

$$S_1 = \frac{(-4m)}{(2\pi)^{\cancel{3}\,2}} \times \cancel{2\pi} \times \int_0^{k_F} |\vec{k}|^2 d|\vec{k}| \int_0^{\pi} \frac{\sin\theta\, d\theta}{2|\vec{k}||\vec{q}|\cos\theta + |\vec{q}|^2 - \tilde{\omega}},$$

stems from φ (azimuthal angle) integration (nothing under the integral depends on φ!)

where we have chosen the z axis to coincide with the direction of $\vec{q}$. After substitution $x = \cos\theta$, we get

$$S_1 = \frac{(-4m)}{(2\pi)^2} \int_0^{k_F} |\vec{k}|^2 d|\vec{k}| \int_{-1}^{1} \frac{dx}{2|\vec{k}||\vec{q}|x + |\vec{q}|^2 - \tilde{\omega}} \Leftrightarrow$$

$$S_1 = -\frac{\cancel{4}^{\,2} m}{(2\pi)^2} \frac{1}{\cancel{2}|\vec{q}|} \underbrace{\int_0^{k_F} d|\vec{k}|\; |\vec{k}| \ln\left(\frac{2|\vec{q}||\vec{k}| + |\vec{q}|^2 - \tilde{\omega}}{-2|\vec{q}||\vec{k}| + |\vec{q}|^2 - \tilde{\omega}}\right)}_{\text{with } a \equiv 2|\vec{q}|,\ b \equiv |\vec{q}|^2 - \tilde{\omega}}$$

We will first determine the indefinite integral

$$\int x \ln\frac{ax+b}{-ax+b}\, dx$$ using integration by parts :

$$\int x \ln\left[\frac{ax+b}{-ax+b}\right] = \int \ln\left[\frac{ax+b}{-ax+b}\right]\left(\frac{x^2}{2}\right)' dx =$$

$$= \frac{x^2}{2}\ln\left[\frac{ax+b}{-ax+b}\right] - \int \frac{-ax+b}{ax+b}\left(\frac{ax+b}{-ax+b}\right)' \frac{x^2}{2}\, dx =$$

$$= \frac{x^2}{2}\ln\left[\frac{ax+b}{-ax+b}\right] - \int \frac{-ax+b}{ax+b}\,\cancel{2}b\,\frac{(-1)}{(-ax+b)^2}(-a)\frac{x^2}{\cancel{2}}\,dx =$$

$$= \frac{x^2}{2}\ln\left[\frac{ax+b}{-ax+b}\right] + \frac{b}{a}\int \frac{x^2}{x^2-\frac{b^2}{a^2}}\,dx =$$

$$= \frac{x^2}{2}\ln\left[\frac{ax+b}{-ax+b}\right] + \frac{b}{a}x + \left(\frac{b}{a}\right)^3 \int \frac{dx}{x^2-\frac{b^2}{a^2}} =$$

$$= \frac{x^2}{2}\ln\left[\frac{ax+b}{-ax+b}\right] + \frac{b}{a}x + \left(\frac{b}{a}\right)^{\cancel{3}\,2} \frac{1}{2\cancel{\frac{b}{a}}}\ln\left[\frac{ax+b}{-ax+b}\right] + C$$

Finally: $\boxed{\int x \ln\left[\frac{ax+b}{-ax+b}\right]dx = \frac{b}{a}x + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln\left[\frac{ax+b}{-ax+b}\right]}$

$\Longrightarrow$ using the last result we easily obtain

$$S_1 = \left(-\frac{2m}{|\vec{q}|}\right)\frac{1}{(2\pi)^2}\left\{\frac{|\vec{q}|^2-\tilde{\omega}}{2|\vec{q}|}k_F + \frac{1}{2}\left[k_F^2 - \frac{(|\vec{q}|^2-\tilde{\omega})^2}{4|\vec{q}|^2}\right] * \right.$$

$$\left. * \ln\left[\frac{2|\vec{q}|k_F + |\vec{q}|^2 - \tilde{\omega}}{-2|\vec{q}|k_F + |\vec{q}|^2 - \tilde{\omega}}\right]\right\}$$

and similarly $(\tilde{\omega} \to -\tilde{\omega})$

$$S_2 = \left(-\frac{2m}{|\vec{q}|}\right)\frac{1}{(2\pi)^2}\left\{\frac{|\vec{q}|^2+\tilde{\omega}}{2|\vec{q}|}k_F + \frac{1}{2}\left[k_F^2 - \frac{(|\vec{q}|^2+\tilde{\omega})^2}{4|\vec{q}|^2}\right] * \right.$$

$$\left. * \ln\left[\frac{2|\vec{q}|k_F + |\vec{q}|^2 + \tilde{\omega}}{-2|\vec{q}|k_F + |\vec{q}|^2 + \tilde{\omega}}\right]\right\}$$

$\Downarrow$

$\mathrm{Re}\,\chi_0^R(\vec{q},\omega)$ can be expressed as

$$\mathrm{Re}\,\chi_0^R(\vec{q},\omega) = -\frac{1}{2}\frac{mk_F}{\pi^2}\left\{1 + \frac{m^2}{k_F|\vec{q}|^3}\left[4\varepsilon_F\varepsilon_{\vec{q}} - (\varepsilon_{\vec{q}}-\omega)^2\right] * \right.$$
$$* \ln\left[\frac{|\vec{q}|v_F + \varepsilon_{\vec{q}} - \omega}{-|\vec{q}|v_F + \varepsilon_{\vec{q}} - \omega}\right]$$

(See Blatt 8, Problem 1!)

$$+ \frac{m^2}{k_F|\vec{q}|^3}\left[4\varepsilon_F\varepsilon_{\vec{q}} - (\varepsilon_{\vec{q}}+\omega)^2\right] *$$
$$\left. * \ln\left[\frac{|\vec{q}|v_F + \varepsilon_{\vec{q}} + \omega}{-|\vec{q}|v_F + \varepsilon_{\vec{q}} + \omega}\right]\right\},$$

where $\varepsilon_{\vec{q}} \equiv \frac{|\vec{q}|^2}{2m}$ and $\frac{mk_F}{\pi^2} \equiv \frac{3n}{2\varepsilon_F} = N_0$ is the density-of-states at the Fermi energy in a non-interacting Fermi gas.

Let us first remind ourselves that the only type of excitations in a non-interacting Fermi gas are the particle-hole pairs. We often say that the response of a non-interacting Fermi gas to density perturbations is incoherent, since it consists of independent particle-hole pairs. In what follows, we'll show that the response changes qualitatively when interactions are turned on – then in addition to particle-hole-type excitations we get collective excitations!

Let us find $\mathrm{Im}\,\chi^{RPA}(\vec{q},\omega)$:

$$\chi^{RPA}(\vec{q},\omega) = \frac{\chi_0^R(\vec{q},\omega)}{1 - V_q\chi_0^R(\vec{q},\omega)} = \frac{\mathrm{Re}\,\chi_0^R + i\,\mathrm{Im}\,\chi_0^R}{1 - V_q(\mathrm{Re}\,\chi_0^R + i\,\mathrm{Im}\,\chi_0^R)} =$$

$$= \frac{\mathrm{Re}\,\chi_0^R + i\,\mathrm{Im}\,\chi_0^R}{1 - V_q\mathrm{Re}\,\chi_0^R - i\,V_q\mathrm{Im}\,\chi_0^R} = \frac{\mathrm{Re}\,\chi_0^R + i\,\mathrm{Im}\,\chi_0^R}{1 - V_q\mathrm{Re}\,\chi_0^R - i\,V_q\mathrm{Im}\,\chi_0^R} * \frac{1 - V_q\mathrm{Re}\,\chi_0^R + i\,V_q\mathrm{Im}\,\chi_0^R}{1 - V_q\mathrm{Re}\,\chi_0^R + i\,V_q\mathrm{Im}\,\chi_0^R}$$

$$= \frac{(\mathrm{Re}\,\chi_0^R + i\,\mathrm{Im}\,\chi_0^R) * (1 - V_q\mathrm{Re}\,\chi_0^R + i\,V_q\mathrm{Im}\,\chi_0^R)}{(1 - V_q\mathrm{Re}\,\chi_0^R)^2 + (V_q\mathrm{Im}\,\chi_0^R)^2} \Longrightarrow$$

$$\mathrm{Im}\,\chi^{RPA}(\vec{q},\omega) = \frac{\mathrm{Im}\,\chi_0^R(1 - V_q\,\mathrm{Re}\,\chi_0^R) + V_q\,\mathrm{Re}\,\chi_0^R\,\mathrm{Im}\,\chi_0^R}{(1 - V_q\,\mathrm{Re}\,\chi_0^R)^2 + (V_q\,\mathrm{Im}\,\chi_0^R)^2}$$

$$\Rightarrow \boxed{\mathrm{Im}\,\chi^{RPA}(\vec{q},\omega) = \frac{\mathrm{Im}\,\chi_0^R(\vec{q},\omega)}{(1 - V_q\,\mathrm{Re}\,\chi_0^R(\vec{q},\omega))^2 + (V_q\,\mathrm{Im}\,\chi_0^R(\vec{q},\omega))^2} \quad (*)}$$

We found the particle-hole continuum in the noninteracting case from the condition $\mathrm{Im}\,\chi_0^R(\vec{q},\omega) \neq 0$. We can now see that there are two possibilities for $\mathrm{Im}\,\chi^{RPA}(\vec{q},\omega) \neq 0$:

1) $1 - V_q\,\mathrm{Re}\,\chi_0^R(\vec{q},\omega) = 0$; $\underbrace{\mathrm{Im}\,\chi_0^R(\vec{q},\omega) = 0}_{\text{this means that we are outside the particle-hole continuum!}}$

in this case both numerator and denominator in (*) are zero!

2) $1 - V_q\,\mathrm{Re}\,\chi_0^R(\vec{q},\omega) = 0$ but $\underbrace{\mathrm{Im}\,\chi_0^R(\vec{q},\omega) \neq 0}_{\text{i.e., we are inside the particle-hole continuum}}$

We'll now explore the consequences of 1) and 2) in the case of Fermi gas with <u>short-range interaction</u>; an example is furnished by He^3 and more recent ones by <u>ultracold Fermi gases</u>, extensively studied during the last decade.

The interaction (two-body) in these systems is modelled by a <u>hard-core potential</u> $\boxed{V(\vec{r}_1 - \vec{r}_2) \propto \delta(\vec{r}_1 - \vec{r}_2)}$

The <u>Fourier transform</u> of such a potential is $\vec{q}$-independent:

$\boxed{V_{\vec{q}} \to V}$

The condition on the real part of $\chi_0^R(\vec{q},\omega)$ then becomes

$$\boxed{\mathrm{Re}\,\chi_0^R(\vec{q},\omega) = \frac{1}{V}} \; .$$

Let us focus on the regime of low frequencies and long-wavelengths (small $\vec{q}$). By expanding the derived expression for $\mathrm{Re}\,\chi_0^R$ (see Blatt 8, Problem 1!) we find the relation

$$\frac{\omega_q}{|\vec{q}|v_F} \ln\left[\frac{\omega_q + |\vec{q}|v_F}{\omega_q - |\vec{q}|v_F}\right] = 1 + \underbrace{\frac{1}{N_0 V}} \qquad (\#)$$

It is straightforward to check that, for an arbitrary N_0V, the last equation has a solution with linear dispersion:

$$\omega_q = c|\vec{q}|$$

Note that V, being a Fourier transform of the real space potential has dimensions energy x volume, while N_0 has dimensions of density divided by energy; thus N_0V is dimensionless!

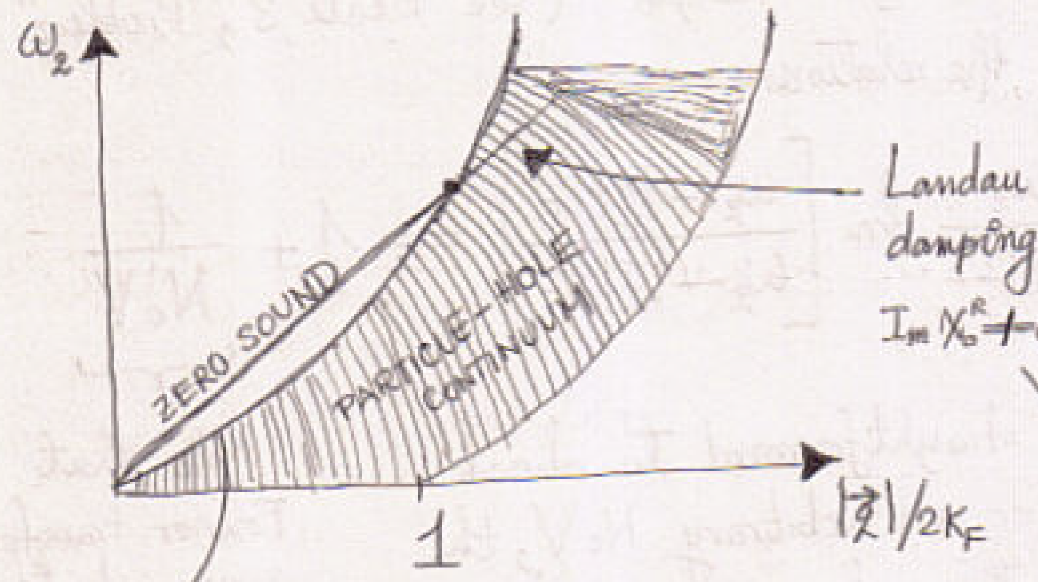
This Goldstone-type solution corresponds to the ZERO-SOUND mode which is gapless and has collective character. When we substitute $\omega_q = c|\vec{q}|$ into Eq. $(\#)$ we obtain an implicit equation for the sound speed c:

$$c = v_F\left(1 + \frac{1}{N_0V}\right)\frac{1}{\ln\left|\frac{c+v_F}{c-v_F}\right|}$$

It is not difficult to show that in the weak interaction limit $\boxed{N_0V \ll 1}$ the last equation can be represented as (see Blatt 8, Problem 1)

$$C \simeq v_F \left(1 + 2 e^{-\frac{1}{N_0 V}}\right) .$$

In particular, in the limit $N_0 V \mapsto 0$ we obtain that $C \to v_F$. This can be illustated as follows

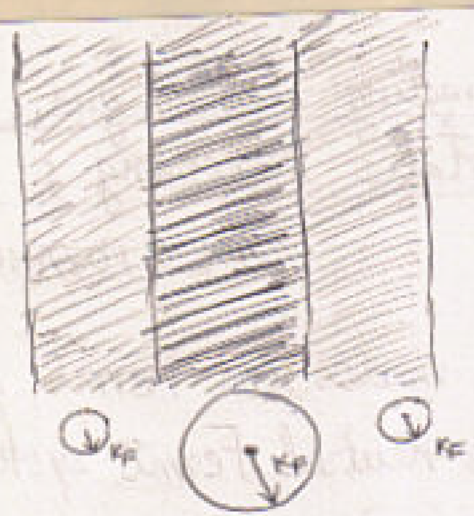


remember that this parabola is given by $\omega_q = \frac{|q|^2}{2m} + v_F|\vec{q}|$, i.e., for small $|\vec{q}|$ it becomes $\omega_q \approx v_F|\vec{q}|$, which is the limiting case of zero-sound with $C \approx v_F$!

the zero-sound collective mode becomes <u>damped</u> inside the particle-hole continuum!

What is the physical meaning of <u>the zero-sound mode</u>?

It should be understood as the "breathing mode" of the Fermi sphere.

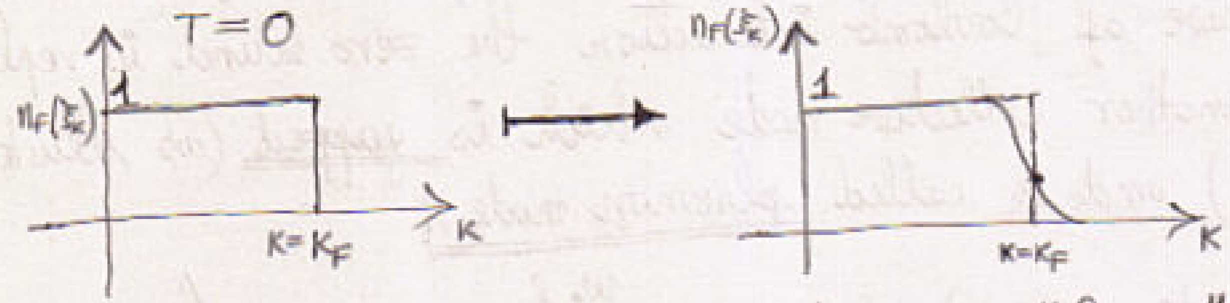


local perturbation in density reflects itself in the "breathing" of the Fermi surface!

he zero-sound mode is a mode related to density fluctuations and it is therefore akin to the ordinary "first") sound! The fingerprints of zero sound are sible in the experiments studying the propagation of und.

It is important to emphasize that zero sound disappears at finite temperatures, or, to be more precise, exists only as a transient!

This is closely related to the thermal smearing that the Fermi distribution undergoes at $T \neq 0$:



As a result, the Fermi surface becomes "fuzzy"!

At the same time, at $T \neq 0$ thermal fluctuations give rise to quasiparticles released from inside the Fermi sphere and the system starts behaving as a gas of classical particles. When the density of such a gas becomes large enough, the ordinary first sound starts propagating!