— THE ELECTRON GAS —

V.M. STOJANOVIĆ

interacting $\bar{e}$ gas, i.e., system of $\bar{e}$s that interact through Coulomb interaction in a uniform, positive charge-neutralizing background

an atom in a metal

nucleus + core $\bar{e}$s $\equiv$ ION

(+Ze)

atom with Z valence $\bar{e}$s

$\Longrightarrow$ ION has charge $+Ze$

ALKALI METALS <u>Li, Na, K, Rb</u> $\rightarrow$ 1 valence $\bar{e}$

e.g. Al $\rightarrow$ 3 valence $\bar{e}$s

The total Hamiltonian of the system of $\bar{e}$s and ions:

$$H = T_{ion} + V_{ion\text{-}ion} + T_{el} + V_{el\text{-}el} + V_{el\text{-}ion}$$

e.g. $$V_{ion\text{-}ion} = \frac{1}{2}\sum_{I,J}\frac{(Ze)^2}{4\pi\varepsilon_0}\frac{1}{|\vec{R}_I - \vec{R}_J|}$$

At $T=0$ the ground-state of the system is a periodic system (lattice!) of ions, with the $\bar{e}$s swirling around — a very complicated system!

What is the <u>MINIMAL MODEL</u> for such a system?

JELLIUM $\longrightarrow$ the discrete nature of the ionic system is approximated by a positively charged, <u>continuous</u> fluid (the ion jellium)

In other words, the charge of ions is smeared out to form a

$$\boxed{V_q \equiv \frac{4\pi e_0^2}{q^2}}$$

$$E^{(2)}_{dir} = \frac{1}{V^2} \sum_{q}{}' \sum_{\substack{\vec{k}_1 \vec{k}_2 \\ \sigma_1 \sigma_2}} \frac{\left(\frac{1}{2} V_q\right)^2}{E^{(0)} - E_n} \left(\theta \ldots \theta \ldots \theta \ldots \theta \ldots\right)$$

behavior at small q

$$V_q^2 \propto \frac{1}{q^4}$$

$$E_0 - E_n \propto k_1^2 + k_2^2 - (k_1+q)^2 - (k_2-q)^2 \underset{q \to 0}{\propto} q$$

$$\sum_{k_1} \ldots \theta(|k_1+q| - k_F)\, \theta(k_F - |k_1|) \overset{?}{\propto} q$$

$$\Rightarrow E^{(2)}_{dir} \propto \int_0 dq\, q^2 \frac{1}{q^4} \frac{1}{q}\, q\, q = \int_0 \frac{dq}{q} = \ln q \Big|_0 \propto \infty$$

The exchange proces does not lead to a divergence since momentum transfer is $\vec{q}$ in the exc. process, but $\vec{k}_2 - \vec{k}_1 - \vec{q}$ in the relaxation $\Rightarrow$

V_q^2 is replaced with $V_q V_{k_2 - k_1 - q} \propto q^{-2}$

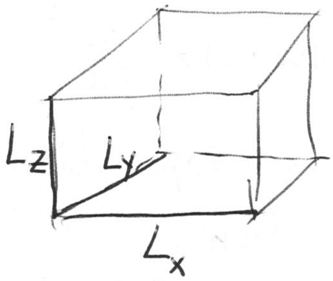
which is less singular than q^{-4}. !

SOLUTION is regularization by taking higher orders of perturbation theory (<u>all orders</u>)

AN ASIDE ...

periodic boundary conditions (P. B. C.) :

$\mathcal{V} = L_x L_y L_z$



$$\Psi(x,y,z) = \Psi(x+L_x, y, z)$$
$$\Psi(x,y,z) = \Psi(x, y+L_y, z)$$
$$\Psi(x,y,z) = \Psi(x, y, z+L_z)$$

$$\Longrightarrow \quad e^{ik_x L_x} = e^{ik_y L_y} = e^{ik_z L_z} = 1$$

$\Longrightarrow$ the momenta are discretized : $\boxed{k_i = \frac{2\pi}{L_i} n_i \quad (i = x,y,z) \quad n_i = 0, \pm 1, \pm 2, \ldots}$

consider a sum in $\vec{k}$-space :

$$\sum_{\vec{k}} f(\vec{k}) = \frac{1}{\frac{(2\pi)^3}{L_x L_y L_z}} \sum_{\vec{k}} f(\vec{k}) \frac{(2\pi)^3}{L_x L_y L_z} = \ldots$$

$$\left\{ \frac{(2\pi)^3}{L_x L_y L_z} = \frac{(2\pi)^3}{\mathcal{V}} \longrightarrow \text{volume of } \vec{k}\text{-space that "belongs" to each permissible } \vec{k} \text{ vector} \right\}$$
$$= \Delta^3 \vec{k}$$

$$\ldots = \frac{\mathcal{V}}{(2\pi)^3} \sum_{\vec{k}} f(\vec{k}) \Delta^3 \vec{k} \xrightarrow[\text{T.D. limit.}]{} \boxed{\frac{1}{\mathcal{V}} \sum_{\vec{k}} f(\vec{k}) = \int \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k})}$$

consider a system with N $\bar{e}$'s :

$$|FS\rangle = C^{\dagger}_{\vec{k}_{N/2},\uparrow} C^{\dagger}_{\vec{k}_{N/2},\downarrow} \ldots C^{\dagger}_{\vec{k}_1,\uparrow} C^{\dagger}_{\vec{k}_1\downarrow} \underbrace{|0\rangle}_{\text{vacuum}}$$

Fermi sea
(-II- sphere)

at every $\vec{k}$ we have $\bar{e}$s of either spin

More rigorous approach (Monte Carlo) shows

$75 < r_s < 100 \longrightarrow$ spin-polarized fluid

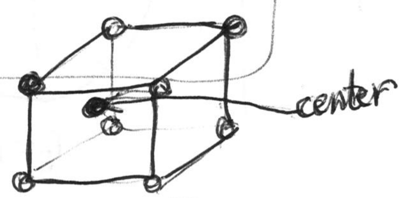
(note the disagreement: 5.45 vs. 75!)

What happens for $r_s > 100$?

$\Longrightarrow$ each e^- gets trapped into potential cage created by the Coulomb repulsion of its neighbors $\Longrightarrow$ Mott insulator phase

an example for

e^- crystalize into a REGULAR lattice: body-centered cubic lattice



$\longrightarrow$ Wigner crystal

{ E.P. Wigner predicted this in 1934 }

restoring force due to positive charge [classical electrost.]

$\propto r^2$

$$N = \langle FS | \sum_{\vec{k},\sigma} n_{\vec{k}\sigma} | FS \rangle \iff$$

$$N = \sum_{\vec{k},\sigma} \langle FS | n_{\vec{k}\sigma} | FS \rangle \overset{!}{=} \sum_{\vec{k},\sigma} \theta(k_F - |\vec{k}|)$$

$\sum_{\vec{k}} \longrightarrow \frac{V}{(2\pi)^3} \int d^3\vec{k}$ and we obtain:

$$N = 2 \frac{V}{(2\pi)^3} \int d^3\vec{k}\, \theta(k_F - |\vec{k}|)$$

spin summation trivially yields factor 2!

integration over angles θ_k, φ_k just yields 4π $\Rightarrow$

$$N = \frac{2V}{8\pi^3} 4\pi \int_0^{+\infty} |\vec{k}|^2 \theta(k_F - |\vec{k}|)\, d|\vec{k}|$$

$$\iff \quad N = \frac{V}{\pi^2} \underbrace{\int_0^{k_F} |\vec{k}|^2 d|\vec{k}|}_{\frac{k_F^3}{3}}$$

$$\iff n = \frac{N}{V} = \frac{k_F^3}{3\pi^2} \Rightarrow \boxed{k_F = (3\pi^2 n)^{1/3}}$$

How about finding the energy per one electron?
(a very similar calculation ...)

We can now separately do this integration over $|\vec{q}|$:

$$\int_0^{2k_F} \left\{ k_F^3\left(1-\frac{|\vec{q}|}{2k_F}\right) + \frac{|\vec{q}|^3}{16}\left(1-\frac{4k_F^2}{|\vec{q}|^2}\right)\right\} d|\vec{q}|$$

$$= 2k_F^4 - \frac{k_F^2}{2}\frac{(2k_F)^2}{2} + \frac{1}{16}\frac{(2k_F)^4}{4} - \frac{k_F^2}{4}\frac{(2k_F)^2}{2}$$

$$= 2k_F^4 - k_F^4 - \frac{k_F^4}{2} + \frac{k_F^4}{4} = \frac{3k_F^4}{4}$$

$$\Downarrow$$

$$\frac{E^{(1)}}{N} = -\frac{\mathcal{V}}{N}\frac{2e_0^2}{(2\pi)^{\cancel{5}\,3}}\cancel{\frac{2*(2\pi)}{3}} * \cancel{2} * \cancel{2\pi} * \cancel{\frac{3k_F^4}{4}} =$$

$$= \boxed{-\frac{\mathcal{V}}{N}\frac{e_0^2}{4\pi^3}k_F^4} \equiv$$

this form of expression for $\frac{E^{(1)}}{N}$ can be useful in problem 1 of Blatt 3!

$$= -\frac{1}{\frac{N}{\mathcal{V}}}\frac{e_0^2}{2a_0}\frac{1}{2\pi^3}k_F^3(k_F a_0)$$

$$n \equiv \frac{N}{\mathcal{V}}, \qquad \frac{e_0^2}{2a_0} \equiv 1\,Ry, \qquad k_F^3 \to 3\pi^2 n, \qquad k_F a_0 \to \left(\frac{9\pi}{4}\right)^{1/3} r_s^{-1}$$

$$\equiv -\frac{1}{\cancel{n}}\frac{3}{2\pi}\cancel{n}\left(\frac{9\pi}{4}\right)^{1/3}\frac{1}{r_s}\,Ry$$

$$= -\left\{\frac{3}{2\pi}\left(\frac{9\pi}{4}\right)^{1/3}\right\}\frac{1}{r_s}\,Ry \approx \boxed{-\frac{0.916}{r_s}\,Ry}$$

* $e^- - e^-$ interactions in standard (Rayleigh-Schrödinger) perturbation theory

inhomogeneous ($\vec{q} \neq 0$) part of the Coulomb interaction:

$$V'_{el\text{-}el} = \frac{1}{2V} \sum_{\vec{k}_1 \vec{k}_2 \sigma_1 \sigma_2} \sum_{\vec{q}}{}' \frac{4\pi e_0^2}{|\vec{q}|^2} c^{\dagger}_{\vec{k}_1+\vec{q},\sigma_1} c^{\dagger}_{\vec{k}_2-\vec{q},\sigma_2} c_{\vec{k}_2 \sigma_2} c_{\vec{k}_1 \sigma_1}$$

$$\boxed{e_0^2 \equiv \frac{e^2}{4\pi\varepsilon_0}}$$

Even more basic question is whether the non-interacting e^- gas can be used as a starting point for a perturbative treatment of the $e^- - e^-$ interaction:

first note that $\frac{E^{(0)}}{N} \equiv E_{KIN} = \frac{3}{5}\varepsilon_F \propto n^{2/3}$

and that $E_{pot} \propto \frac{e^2}{d} = \frac{e^2}{n^{-1/3}} \propto n^{1/3}$,

so that

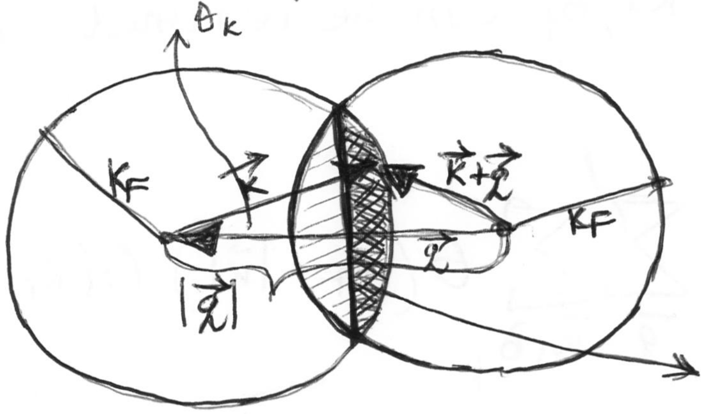
$$\boxed{\frac{E_{pot}}{E_{KIN}} \propto \frac{n^{1/3}}{n^{2/3}} = n^{-1/3} \xrightarrow[(n \to \infty)]{} 0}$$

$\Longrightarrow$ the importance of $e^- - e^-$ interaction diminishes with increasing density, i.e., the kinetic energy becomes

$$\frac{E^{(1)}}{N} = -\frac{\nu}{N}\frac{2e_0^2}{(2\pi)^5}\int\frac{d^3\vec{q}}{|\vec{q}|^2}\underbrace{\int d^3\vec{k}\,\theta(k_F-|\vec{k}+\vec{q}|)\,\theta(k_F-|\vec{k}|)}$$

THIS PART WILL BE SOME FUNCTION OF $|\vec{q}|$ THAT WE NEED TO FIND!

let us use geometrical representation:



the integral is nonzero only for !! $\boxed{|q| < 2K_F}$

Our domain of integration is the shaded area (and our integral is ITS VOLUME!)

This area consists of two identical parts so we need only integrate over the crosshatched domain D and multiply the result by 2!

$$\int d^3\vec{k}\,\theta(k_F-|\vec{k}+\vec{q}|)\,\theta(k_F-|\vec{k}|) = 2*\int_{D}\theta(k_F-|\vec{k}+\vec{q}|)\,\theta(k_F-|\vec{k}|)\,d^3\vec{k}$$

this function under integral does depend on angles!!

HOW TO DESCRIBE IT?

$$\frac{|\vec{q}|}{2K_F} \leq \cos\theta_K \leq 1 \quad \left[\text{equivalently: } 0 \leq \theta_K \leq \cos^{-1}\left(\frac{|\vec{q}|}{2K_F}\right)\right]$$

$\underline{1^{st} \text{ order P.T.}}$ remember $H = H_0 + V$

$$\boxed{E^{(1)} = \langle 0|V|0\rangle},\text{ where}$$

$|0\rangle$ is the ground-state of H_0

our ground state is $|FS\rangle$! $\Rightarrow$

$$\boxed{\frac{E^{(1)}}{N} = \frac{1}{N}\langle FS|V'_{el\text{-}el}|FS\rangle}$$

$$\frac{E^{(1)}}{N} = \frac{1}{2\nu N}\sum_{\substack{\vec{k}_1,\vec{k}_2\\ \sigma_1,\sigma_2}}\sum_{\vec{q}}{}' \frac{4\pi e_0^2}{|\vec{q}|^2} * \text{M.E.}$$

this is what we want to find first:

$$\langle FS|\, c^\dagger_{\vec{k}_1+\vec{q},\sigma_1} c^\dagger_{\vec{k}_2-\vec{q},\sigma_2} c_{\vec{k}_2\sigma_2} c_{\vec{k}_1\sigma_1}|FS\rangle$$

$|\vec{k}_1|, |\vec{k}_2| \longrightarrow$ (see FIGURE 1)

two creation operators should bring us back to $|FS\rangle$!

$\vec{k}_1+\vec{q} = \vec{k}_1$, $\vec{k}_2-\vec{q} = \vec{k}_2$ implies $\vec{q} = 0$, but this was already excluded!

(DIRECT PROCESSES)

$\Rightarrow$ $\boxed{\vec{k}_1+\vec{q} = \vec{k}_2,\ \sigma_1 = \sigma_2}$ is the only possibility!

$\Rightarrow$ important to remember:

in the 1^{st} order only exchange processes contribute!

$$\Rightarrow \quad M.E. =$$

$$= \langle FS | C^{+}_{\vec{k}_1+\vec{q},\sigma_1} C^{+}_{\vec{k}_2-\vec{q},\sigma_2} C_{\vec{k}_2\sigma_2} C_{\vec{k}_1\sigma_1} | FS \rangle =$$

$$= \delta_{\vec{k}_2,\vec{k}_1+\vec{q}}\, \delta_{\sigma_1\sigma_2} \langle FS | C^{+}_{\vec{k}_1+\vec{q},\sigma_1} C^{+}_{\vec{k}_1\sigma_1} C_{\vec{k}_1+\vec{q},\sigma_1} C_{\vec{k}_1\sigma_1} | FS \rangle$$

note that (anticommutation):

$$C^{+}_{\vec{k}_1\sigma_1} C_{\vec{k}_1+\vec{q},\sigma_1} = - C_{\vec{k}_1+\vec{q},\sigma_1} C^{+}_{\vec{k}_1\sigma_1}$$

$$\Downarrow$$

$$M.E. = -\delta_{\vec{k}_2,\vec{k}_1+\vec{q}}\, \delta_{\sigma_1\sigma_2} \langle FS | n_{\vec{k}_1+\vec{q},\sigma_1} n_{\vec{k}_1\sigma_1} | FS \rangle$$

remember that $n_{\vec{k}_1\sigma_1} | FS \rangle = \theta(k_F - |\vec{k}_1|) | FS \rangle$

then $\boxed{n_{\vec{k}_1+\vec{q},\sigma_1} n_{\vec{k}_1\sigma_1} | FS \rangle = \theta(k_F - |\vec{k}_1+\vec{q}|)\, \theta(k_F - |\vec{k}_1|) | FS \rangle}$

Thus we have:

[redacted] $\langle FS | c^{\dagger}_{\vec{k}_1+\vec{q},\sigma_1} c^{\dagger}_{\vec{k}_2-\vec{q}\sigma_2} c_{\vec{k}_2\sigma_2} c_{\vec{k}_1\sigma_1} | FS \rangle$

$$= -\delta_{\vec{k}_2,\vec{k}_1+\vec{q}}\,\delta_{\sigma_1\sigma_2}\,\theta(k_F - |\vec{k}_1+\vec{q}|)\,\theta(k_F-|\vec{k}_1|)$$

kill sums over $\vec{k}_2, \sigma_2$;

$\vec{k}_1, \sigma_1$ can be renamed into $\underline{\underline{\vec{k}, \sigma}}$!

$\Downarrow$

$$\frac{E^{(1)}}{N} = -\frac{1}{2\nu N}\sum_{\vec{q}}{}' \sum_{\vec{k},\sigma} \underbrace{\theta(k_F-|\vec{k}|)\,\theta(k_F-|\vec{k}+\vec{q}|)\frac{4\pi e_0^2}{|\vec{q}|^2}}_{\text{nothing depends on the spin } \sigma\text{!}}$$

$\sum_{\sigma}$ only yields a factor of 2!

$\Downarrow$

$$\frac{E^{(1)}}{N} = -\frac{1}{\nu N}\sum_{\vec{q}}{}' \frac{4\pi e_0^2}{|\vec{q}|^2}\sum_{\vec{k}} \theta(k_F-|\vec{k}+\vec{q}|)\,\theta(k_F-|\vec{k}|)$$

we can now turn the sums into integrals

$$\sum_{\vec{q}}(\ldots) \to \frac{\nu}{(2\pi)^3}\int d^3\vec{q}\,(\ldots) \qquad \sum_{\vec{k}}(\ldots) \to \frac{\nu}{(2\pi)^3}\int d^3\vec{k}.$$

$$\Rightarrow \frac{E^{(1)}}{N} = -\frac{\nu}{N}\frac{4\pi e_0^2}{(2\pi)^6}\int \frac{d^3\vec{q}}{|\vec{q}|^2}\int d^3\vec{k}\,\theta(k_F-|\vec{k}+\vec{q}|)\,\theta(k_F-|\vec{k}|)$$

Let us establish some relevant length and energy scales:

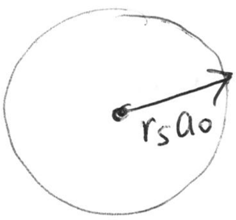
$$\boxed{a_0 \equiv \frac{\hbar^2}{me^2} \cong 0.53\,\text{Å}} \quad \text{(Bohr radius)}$$

$$E_0 = -\frac{e_0^2}{2a_0} = -13.6\,\text{eV}$$

$$\Longrightarrow \quad \boxed{1\,\text{Ry} = \frac{e_0^2}{2a_0} = 13.6\,\text{eV}}$$

Rydberg
- atomic energy unit

Finally, dimensionless parameter:



$$\frac{4}{3}(r_s a_0)^3 \pi = \frac{V}{N} = \frac{1}{n}$$

$$\Longleftrightarrow \quad \frac{4}{3}(r_s a_0)^3 \pi = \frac{3\pi^2}{3\pi^2 n} = \frac{3\pi^2}{k_F^3}$$

$$\Longleftrightarrow \quad \boxed{r_s = \left(\frac{9\pi}{4}\right)^{1/3} (k_F a_0)^{-1}}$$

or $\boxed{r_s \cong \frac{1.92}{k_F a_0}}$

metallic densities of conduction e^-s

$$\longleftrightarrow \quad \boxed{r_s = 2-6}$$

In units of $\underbrace{r_s, \text{Ry}}$: $\boxed{\frac{E^{(0)}}{N}} = \frac{3}{5}\left(\frac{9\pi}{4}\right)^{2/3} \frac{e_0^2}{2a_0} r_s^{-2} \simeq \boxed{\frac{2.21}{r_s^2}\,\text{Ry}}$

For each fixed θ_K, $|\vec{k}|$ goes from

$$\frac{|\vec{q}|}{2\cos\theta_K} \text{ to } k_F$$

$$\Rightarrow \int d^3\vec{k}\, \theta\left(k_F - |\vec{k}+\vec{q}|\right) \theta\left(k_F - |\vec{k}|\right) =$$

$$= 2 * \underbrace{2\pi}_{\substack{\text{integration} \\ \text{over } \varphi_K}} \int_0^{\cos^{-1}\left(\frac{|\vec{q}|}{2k_F}\right)} \sin\theta_K\, d\theta_K * \underbrace{\int_{\frac{|\vec{q}|}{2\cos\theta_K}}^{k_F} |\vec{k}|^2 d|\vec{k}|}_{\frac{1}{3}\left[k_F^3 - \left(\frac{|\vec{q}|}{2\cos\theta_K}\right)^3\right]}$$

$$= \frac{4\pi}{3} \int_{\frac{|\vec{q}|}{2k_F}}^{1} \left[k_F^3 - \left(\frac{|\vec{q}|}{2u}\right)^3\right] du$$

where we introduced new variable $\boxed{\cos\theta_K \equiv u}$

$$= \frac{4\pi}{3} \left\{ k_F^3 \left(1 - \frac{|\vec{q}|}{2k_F}\right) + \frac{|\vec{q}|^3}{8\cdot 2}\left(1 - \frac{4k_F^2}{|\vec{q}|^2}\right) \right\}$$

$$\Downarrow$$

$$\frac{E^{(1)}}{N} = -\frac{\nu}{N} \frac{2e_0^2}{(2\pi)^5} * \frac{4\pi}{3} * \underbrace{4\pi}_{\text{integration over } \vec{q} \text{ angles!}} *$$

$$* \int_0^{2k_F} \frac{\cancel{|\vec{q}|^2}}{\cancel{|\vec{q}|^2}} d|\vec{q}| \left\{ \dots \right\}$$

$$E^{(0)} = \langle FS | H_{el} | FS \rangle = \sum_{\vec{k},\sigma} \frac{\hbar^2 k^2}{2m} \underbrace{\langle FS | n_{k\sigma} | FS \rangle}_{\text{once again!}}$$

$$\Rightarrow \quad E^{(0)} = 2 \frac{V}{(2\pi)^3} \int d^3\vec{k}\, |\vec{k}|^2\, \theta(k_F - |\vec{k}|)$$

from spin!

$$\Downarrow$$

$$E^{(0)} = \frac{\cancel{2}V}{\cancel{8}\pi^{\cancel{3}2}} * \cancel{4\pi} * \frac{\hbar^2}{2m} \int_0^{+\infty} |\vec{k}|^4\, \theta(k_F - |\vec{k}|)\, d|\vec{k}|$$

$$\Rightarrow \quad E^{(0)} = \frac{V}{\pi^2} \frac{k_F^5 \cdot \hbar^2}{5\; 2m} = \frac{V}{\pi^2} \underbrace{\frac{\hbar^2 k_F^2}{2m}}_{\mathcal{E}_F} \cdot \underbrace{k_F^3}_{3\pi^2 n} \cdot \frac{1}{5} \quad \Leftrightarrow$$

$$E^{(0)} = \frac{Vn}{\cancel{\pi^2}} \cdot 3\cancel{\pi^2} \frac{\mathcal{E}_F}{5} = N \cdot \frac{3\mathcal{E}_F}{5} \quad \Leftrightarrow$$

$$\boxed{\frac{E^{(0)}}{N} = \frac{3}{5}\mathcal{E}_F}$$

<u>typical numbers</u>: $n \sim 10^{28}\, m^{-3}$

$\mathcal{E}_F \sim (1-10)\, eV$

Fermi temperature

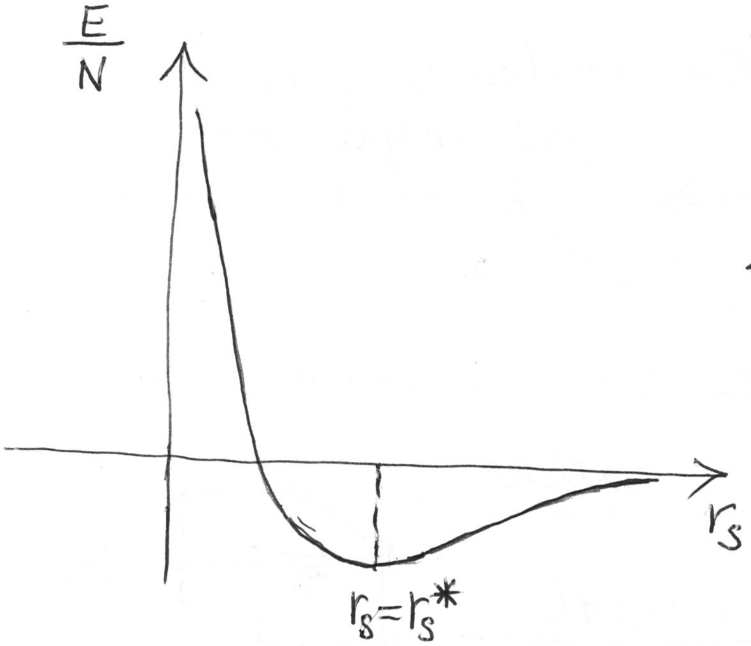
$k_B T_F = \mathcal{E}_F \Rightarrow T_F \sim 10^4 - 10^5 K$ <u>very high</u> temperature!

remember that $\boxed{k_B T \quad \sim 25.5\, meV}$ $\boxed{\lambda_F \sim 0.1\, nm}$

$$\frac{E}{N} \underset{\substack{r_s \to 0 \\ (\text{high-density}!)}}{=\!=} \frac{E^{(0)}+E^{(1)}}{N} = \boxed{\left(\frac{2.211}{r_s^2} - \frac{0.916}{r_s}\right) R_y}$$

WHY IS THE EXCHANGE CONTRIBUTION NEGATIVE?
PAULI PRINCIPLE for $\bar{e}$s in real space!

exchange contribution to Coulomb interaction takes care of two things:



1) $\frac{E}{N}$ has a minimum at a finite value of r_s!

2) the energy value $\frac{E}{N}$ in this minimum is <u>negative</u>!

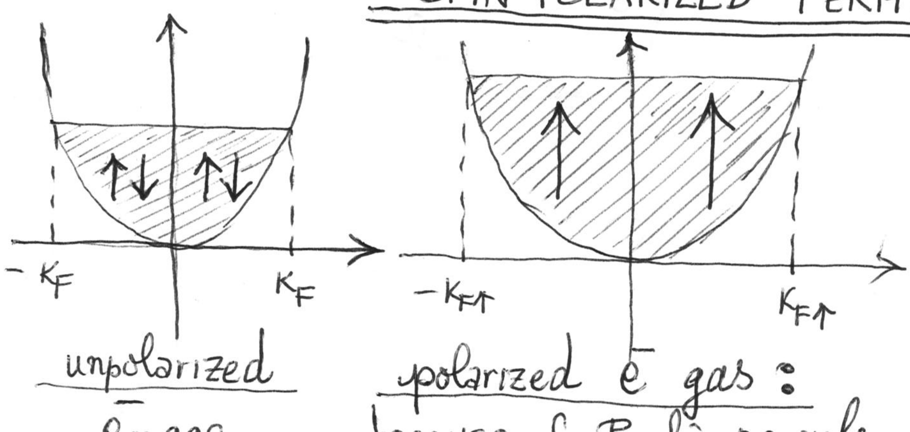
negative energy $\Longrightarrow$ <u>no</u> external trapping potentials are needed to stabilize the system!

$$\frac{d}{dr_s}\left[\frac{E}{N}\right]\Bigg|_{r_s = r_s^*} = 0 \Longrightarrow \boxed{r_s^* \simeq 4.83} \Rightarrow \boxed{\frac{E^*}{N} = -1.29\,\text{eV}}$$

experiment on Na : $r_s^* \approx 3.96$, $\frac{E^*}{N} = -1.13\,\text{eV}$

THE AGREEMENT IS ONLY QUALITATIVE!

SPIN-POLARIZED FERMI GAS

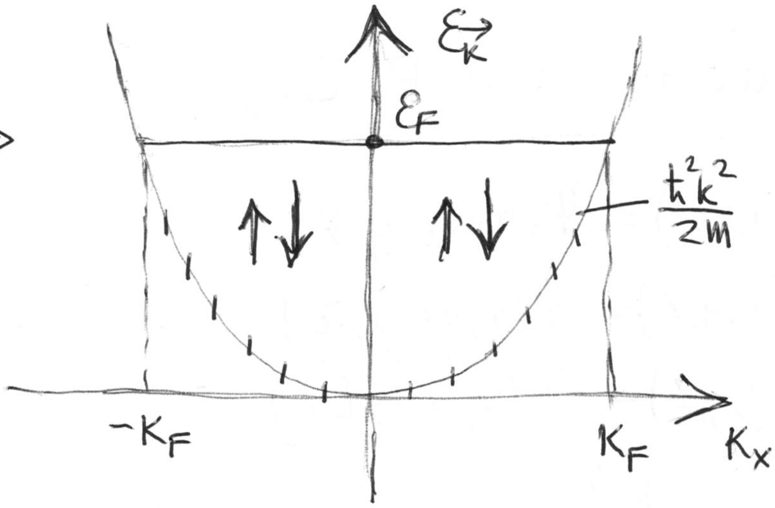


for the non-interacting $\bar{e}$-gas this polarized state can never be stable, as it always would have a higher energy than the unpolarized one!

It is apriori not obvious what happens in the presence of interaction

1^{st} order P.T. $\Longrightarrow$ for $\boxed{r_s \gtrsim 5.45}$ (i.e. sufficiently

$\Longrightarrow$



The energy of the topmost level is the Fermi energy $\mathcal{E}_F$; the corresponding momentum

$$k_F \equiv \frac{1}{\hbar}\sqrt{2m\mathcal{E}_F}$$

is the Fermi momentum.

Fermi wavelength

$$\lambda_F \equiv \frac{2\pi}{k_F}$$

Fermi velocity $v_F \equiv \frac{\hbar k_F}{m}$

WHAT IS $n_{\vec{k}\sigma}|FS\rangle$?

$$n_{\vec{k}\sigma}|FS\rangle = \begin{cases} |FS\rangle, & \text{for } |\vec{k}| < k_F \\ 0, & \text{otherwise} \end{cases}$$

This can be written as: θ – Heaviside f-on

$$n_{\vec{k}\sigma}|FS\rangle = \theta(k_F - |\vec{k}|)\,|FS\rangle$$

$$\Longrightarrow \quad \boxed{\langle FS|\,n_{\vec{k}\sigma}\,|FS\rangle = \theta(k_F - |\vec{k}|)}$$

We'll use this result multiple times in what follows!

$\Longrightarrow$ Let us first establish the relation between

2^{nd} order perturb. theory

WE EXPECT TO GET A RESULT BETTER THAN IN 1^{st} ORDER, BUT...

$$\frac{E^{(2)}}{N} = \frac{1}{N} \sum_{|n\rangle \neq |FS\rangle} \frac{\langle FS | V'_{e-e} | n \rangle \langle n | V'_{e-e} | FS \rangle}{E^{(0)} - E_n}$$

all the intermediate states $|n\rangle$ must be different from $|FS\rangle$!

momentum-cons. Coulomb interaction $\Rightarrow$ interm. states where two particles are injected out of the FS

See $\boxed{\text{FIGURE 2}}$! (Pauli principle $\rightarrow$ no empty states!)

From such an intermediate state, $|FS\rangle$ is restored by putting 2e back to ~~their~~ holes ~~they~~ they left behind

direct: each e^- goes to "the same hole"

exchange: each e^- goes to "the hole left by the other e^-"!

direct processes lead to divergence because of the beh. of Coulomb interaction at small momentum transfer!

$$\Delta\ |n\rangle = \theta(|\vec{k}_1+\vec{q}| - k_F)\,\theta(|\vec{k}_2-\vec{q}| - k_F)\,\theta(k_F - |\vec{k}_1|)\,\theta(k_F - |\vec{k}_2|)\, c^\dagger_{\vec{k}_1+\vec{q}\,\sigma_1} c^\dagger_{\vec{k}_2-\vec{q}\,\sigma_2} c_{\vec{k}_2\sigma_2} c_{\vec{k}_1\sigma_1} |FS\rangle$$

direct pr.

the same momentum transfer $\vec{q}$ must be involved in

STATIC $\longrightarrow$ the kinetic energy of ions should be neglected (IMPLICIT ASSUMPTION)

$$T_{ion} \approx 0$$

Moreover, if we focus on the non-interacting case, then the homogeneous part of the e.g. together with ion jellium forms a completely charge-neutral system:

$$V_{ion-ion} + V_{el-el}^{(hom.)} + V_{el-ion} = 0$$

$\Rightarrow$ the original Hamiltonian reduces to

$$\boxed{H_{jel} = T_{el} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N} \nabla_j^2}$$

Hamiltonian of the free $\bar{e}$-gas!

single particle states are plane-wave-like:

$$\psi_{\vec{k}\sigma} = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}} \chi_\sigma \quad (\sigma = \uparrow, \downarrow)$$

$$H_{jel}\,\psi_{\vec{k},\sigma} = \frac{\hbar^2 k^2}{2m}\psi_{\vec{k},\sigma} \Rightarrow$$ H_{jel} in the second quantization:

$$H_{jel} = \sum_{\vec{k},\sigma} \frac{\hbar^2 k^2}{2m} \underbrace{c^{+}_{\vec{k}\sigma} c_{\vec{k}\sigma}}_{n_{\vec{k}\sigma}}$$

particle-number

$c_{\vec{k}\sigma}$ ($c^{+}_{\vec{k}\sigma}$) destroys (creates) an $\bar{e}$ with momentum $\vec{k}$ and spin σ!!