

Detection of qubit-oscillator entanglement in nanoelectromechanical systems

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Experiments over the past years have demonstrated that it is possible to bring nanomechanical resonators and superconducting qubits close to the quantum regime and to measure their properties with an accuracy close to the Heisenberg uncertainty limit. Therefore, it is just a question of time before we will routinely see true quantum effects in nanomechanical systems. One of the hallmarks of quantum mechanics is the existence of entangled states. We propose a realistic scenario making it possible to detect entanglement of a mechanical resonator and a qubit in a nanoelectromechanical setup. The detection scheme is all done by standard current and noise measurements of an atomic point contact coupled to an oscillator and a qubit. This setup could allow for the first observation of entanglement between a continuous and a discrete quantum system in the solid state.

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In recent years, nanoelectromechanical systems (NEMS) have become a strong focus of research in theoretical and experimental physics [1]. One of the practical reasons for this development is the prospective use of NEMS to design devices which allow the measurement of position, force and mass [2–4] with unprecedented accuracies. From a more fundamental point of view, NEMS operate at the boundary between the classical and the quantum world and recent works have suggested that NEMS will soon allow the observation of quantum mechanical states in mesoscopic mechanical systems [5].

The observation of quantum states of matter in such systems generally requires ultralow temperature and low dissipation. Bringing a nanomechanical oscillator of frequency Ω near its ground state means reaching a temperature $T \ll \hbar\Omega/k_B$. Various schemes to cool an oscillator to its ground state have been proposed [6–8], and experiments on nanomechanical systems are now approaching this limit [9, 10]. Moreover, high quality factors have been achieved which lead to relaxation and decoherence times long enough for the measurement of quantum states [11, 12].

One of the most rewarding endeavors involves the creation and detection of non-classical correlations (entanglement) between the nanomechanical oscillator and another quantum system. The easiest option would be to entangle the oscillator with a mesoscopic system whose properties are well understood and in which quantum effects can routinely be observed: a superconducting qubit [13, 14]. It has been demonstrated that these devices have decoherence times which can exceed oscillation periods of nanomechanical resonators by several orders of magnitude [15]. Various theoretical proposals have been made on how entanglement between an oscillator and a qubit can be created [16, 17] and such systems have been successfully coupled in experiments [5, 18].

In this article, we propose a system which allows the detection of entanglement between an oscillator and a qubit using an electronic measurement in an atomic point contact (APC). The electronic system is based on a tunnelling contact, a read-out device which is known to be quantum-limited [19]. We find that the measurement of the current and the symmetrised current noise in this system allows the evaluation of a criterion for entanglement [20] based on the density matrix of the oscillator-qubit system. This allows for the detection of entanglement in *arbitrary* pure or mixed states.

The system we investigate consists of a nanomechanical oscillator, a qubit and a biased APC. Both the oscillator and the qubit are coupled to the APC and thus modulate its transmission coefficient. The APC consists of two electron reservoirs (“left” and “right”) at chemical potentials $\mu_{L,R}$ which are subject to a voltage difference $V = \mu_L - \mu_R$. The Hamiltonian of the APC reads (using units $e = \hbar = 1$)

$$H_{el} = \sum_{\alpha=R,L} \sum_k (\epsilon_k + \mu_\alpha) \psi_{\alpha,k}^\dagger \psi_{\alpha,k} ,$$

$$H_T = \gamma Y \psi_L^\dagger(z=0) \psi_R(z=0) + \text{h.c.} , \quad (1)$$

where $\psi_\alpha^\dagger(z) = (1/\sqrt{L}) \sum_k e^{-ikz} \psi_{\alpha,k}^\dagger$ ($\alpha = L, R$) creates an electron at position z in the left/right reservoir, respectively. As a simplification, we assume a constant density of states $\rho_0 = 1/(\pi v_F)$ where v_F is the Fermi velocity. In the tunnelling Hamiltonian H_T , the counting operator Y (Y^\dagger) decreases (increases) the transferred charge by one, $Y |n_R + 1\rangle = |n_R\rangle$. The corresponding number operator is defined by $\hat{n}_R |n_R\rangle = n_R |n_R\rangle$. The tunnelling amplitude γ will be specified shortly.

The oscillator and the qubit are described by the

Hamiltonians

$$\begin{aligned} H_R &= \frac{p^2}{2m} + \frac{1}{2}m\Omega^2 x^2, \\ H_Q &= \epsilon\sigma_z + \Delta\sigma_x, \end{aligned} \quad (2)$$

where m and Ω denote the effective mass and the frequency of the oscillator, respectively. In the qubit Hamiltonian, $\sigma_{x,y,z}$ denote the Pauli matrices. For $\Delta = 0$, the energy difference between the two qubit states is given by 2ϵ . A finite Δ enables tunnelling between the states.

The state of the qubit-oscillator system modulates the tunnelling amplitude γ of the APC. If the oscillator acts as one of the electron reservoirs of the APC [21] as shown in Fig. 1, the tunnelling gap depends on the oscillator displacement x . For small x one obtains $\gamma \propto \gamma_0 + \gamma_1 x$. The same dependence can also be realised for capacitive coupling [22]. The qubit can be realised as a Cooper pair box in which case a depletion of the electron reservoirs of the APC depending on the state of the qubit leads to an additional term $\gamma_2\sigma_z$ in the tunnelling amplitude. Irrespective of the concrete realisation, to lowest order the combined effect of the oscillator and the qubit leads to

$$\gamma = \gamma_0 + \gamma_1 x + \gamma_2\sigma_z. \quad (3)$$

In general, the amplitudes $\gamma_j = |\gamma_j|e^{i\delta_j}$ ($j = 0, 1, 2$) can be complex. Since the global phase is irrelevant, we set $\delta_0 = 0$. Finite phases $\delta_{1,2}$ can be realised experimentally by closing the electric circuit using an additional tunnel junction as shown in Fig. 1 [23]. Threading the loop with a magnetic flux causes Aharonov-Bohm phases which can be absorbed in the tunnelling amplitudes and generally lead to finite phases δ_1 and δ_2 . This is discussed in more detail in [24]. The benefits of a controllable δ_1 have been investigated for a system consisting of an APC and an oscillator [23]: while for $\delta_1 = 0$, the current noise only depends on the oscillator position $\langle x^2 \rangle$, a finite δ_1 leads to terms proportional to $\langle p^2 \rangle$ and thus contains information about the oscillator momentum. Similarly, the presence of tunable phases $\delta_{1,2}$ increases the number of measurable oscillator and qubit properties.

The current operator is defined by $I = \frac{d}{dt}\hat{n}_R$. Our main objective will be the calculation of the average current $\langle I \rangle$ and the symmetrised noise spectral density,

$$S(\omega) = \frac{1}{2} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle\langle \{I(\tau), I(0)\} \rangle\rangle, \quad (4)$$

where $\langle\langle I(\tau)I(0) \rangle\rangle = \langle I(\tau)I(0) \rangle - \langle I(\tau) \rangle \langle I(0) \rangle$ and $\{\cdot, \cdot\}$ denotes the anti-commutator. We shall find that the average current as well as the noise depend on expectation values of products of qubit and oscillator operators. From these expectation values, the bipartite expectation value matrix (EVM) can be constructed. It is a complex 6×6 -

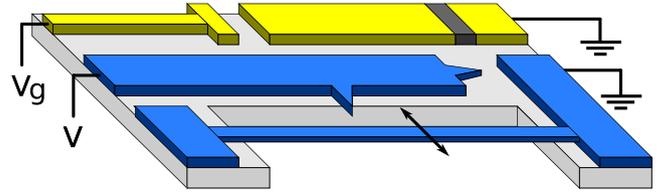


FIG. 1: (Color online) Possible schematic picture of qubit and oscillator coupled to an APC. The APC is biased with a voltage V . The oscillator is coupled to an STM tip with position-dependent tunnelling amplitude γ . The qubit is realised as a Cooper pair box electrostatically coupled to the junctions. A magnetic flux Φ is applied in order to obtain phase shifts in the tunnelling amplitudes.

matrix defined by [20]

$$\chi = \begin{pmatrix} \langle\langle \uparrow\uparrow | \uparrow\uparrow \otimes B \rangle\rangle & \langle\langle \uparrow\uparrow | \downarrow\downarrow \otimes B \rangle\rangle \\ \langle\langle \downarrow\downarrow | \uparrow\uparrow \otimes B \rangle\rangle & \langle\langle \downarrow\downarrow | \downarrow\downarrow \otimes B \rangle\rangle \end{pmatrix}, \quad (5)$$

where

$$B = \begin{pmatrix} 1 & x & p \\ x & x^2 & S_{xp} \\ p & S_{xp} & p^2 \end{pmatrix} \quad (6)$$

and we used $S_{xp} = \frac{1}{2}(xp + px)$. All expectation values are taken with respect to the qubit-oscillator state described by the density matrix $\rho_{q,osc}$, e.g. $\langle\langle \uparrow\uparrow | \uparrow\uparrow x \rangle\rangle = \text{Tr}[\rho_{q,osc} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| x]$. It has been shown [20] that for any separable state $\rho_{q,osc}$,

$$\chi \pm \frac{i}{2}\rho_q \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \geq 0, \quad (7)$$

i.e. both matrices must be positive semidefinite. Here, $\rho_q = \text{Tr}_{osc} \rho_{q,osc}$ denotes the reduced density matrix of the qubit. Once the complete EVM is known, a violation of Eq. (7) proves that $\rho_{q,osc}$ is an entangled state. In the following, we show that the current and its noise contain enough information to construct the EVM and thus deduce entanglement of the oscillator and the qubit. Hence, this measurement provides a separability criterion for an oscillator and a qubit, comparable to a Bell inequality measurement which provides a separability criterion for two qubits.

Since the oscillator-qubit state is not necessarily stationary, both the average current and the noise spectrum will in general be time-dependent. Therefore, we relate the current and noise at time t to the expectation values of the qubit-oscillator system taken at the same time. We calculate the current using perturbation theory in the tunnelling Hamiltonian H_T and using $H_0 = H_{el} + H_R + H_Q$ as the unperturbed Hamiltonian. The method is discussed in more detail in [24].

Calculating the time-dependence of the qubit and oscillator operators using the Hamiltonian H_0 , the Kubo

formula straightforwardly yields the average current,

$$\begin{aligned}
\langle I \rangle &= 2VT_0 + \frac{V}{2}T_2 + \frac{V}{2}T_1 \langle \hat{x}^2 \rangle \\
&+ \sqrt{T_0 T_1} [2V \cos(\delta_1) \langle \hat{x} \rangle + \Omega \sin(\delta_1) \langle \hat{p} \rangle] \\
&+ 2\sqrt{T_0 T_2} [V \cos(\delta_2) \langle \sigma_z \rangle + \Delta \sin(\delta_2) \langle \sigma_y \rangle] \\
&+ \sqrt{T_1 T_2} \left[V \cos(\delta_{21}) \langle \hat{x} \sigma_z \rangle - \frac{\Omega}{2} \sin(\delta_{21}) \langle \hat{p} \sigma_z \rangle \right. \\
&\left. + \Delta \sin(\delta_{21}) \langle \hat{x} \sigma_y \rangle \right] + I_{0,\text{osc}} + I_{0,\text{q}} , \quad (8)
\end{aligned}$$

where we used $\delta_{21} = \delta_2 - \delta_1$. The dimensionless conductance of the bare tunnel junction is given by $T_0 = \pi \rho_0^2 |\gamma_0|^2$. Moreover, using the relaxation rates for the oscillator and the qubit [23, 25],

$$\begin{aligned}
\Gamma_1 &= 2\pi \rho_0^2 |\gamma_1|^2 / m , \\
\Gamma_2 &= 4\pi \rho_0^2 V |\gamma_2|^2 , \quad (9)
\end{aligned}$$

we defined dimensionless conductances for the oscillator and the qubit by $T_1 = \Gamma_1 / \Omega$ and $T_2 = \Gamma_2 / V$. The oscillator displacement is measured in units of its zero point motion $x_0 = 1/\sqrt{2m\Omega}$, i.e. $\hat{x} = x/x_0$ and $\hat{p} = 2x_0 p$. Finally, $I_{0,\text{osc}} = -\frac{1}{2}T_1 \text{sgn}(V) \min(|V|, \Omega)$ is the current due to the zero-point fluctuations of the oscillator. A similar term arises for the qubit, $I_{0,\text{q}}(|V| > 2\alpha) = T_2 \Delta \langle \sigma_x \rangle \text{sgn}(V)$ while $I_{0,\text{qubit}}(|V| < 2\alpha) = T_2 V \langle \sigma_x \rangle \Delta / (2\alpha)$ where $\alpha = \sqrt{\epsilon^2 + \Delta^2}$.

For $\gamma_2 = 0$, the result coincides with the known result for a system containing only the oscillator coupled to an APC [23], while for $\gamma_1 = \delta_2 = 0$, the known result for an APC coupled to a qubit emerges [25]. In the general case, a current measurement enables us to deduce the qubit expectation values needed for the construction of the reduced density matrix ρ_q . However, the correlation functions contained in the result for the current are insufficient to construct the EVM.

The calculation of the noise spectral density can most easily be accomplished by using the Redfield equation in connection with the Born-Markov approximation. For this purpose, we split the complete system into a fermionic ‘‘bath’’, $H_B = H_{el}$ and the actual ‘‘system’’, $H_S = H_Q + H_R$, coupled by H_T . The bath timescale can be estimated to be of order $1/V$ since $eV/(2\pi\hbar)$ corresponds to the attempt frequency at which electrons arrive at the tunnel junction. In the limit $V \gg \Omega, \Delta, \epsilon$, we can assume the bath timescale to be much faster than the system timescales. To lowest nonvanishing order in the tunnelling, the equation of motion for the reduced system density matrix $\rho_S(t) = \text{Tr}_B \rho(t)$ then reads

$$\begin{aligned}
\dot{\rho}_S(t) &= -i[H_S, \rho_S(t)] \\
&- \int_0^\infty d\tau \text{Tr}_B [H_T, [H_T(-\tau), \rho_S(t) \otimes \rho_B]] . \quad (10)
\end{aligned}$$

This equation can be used to calculate expectation values of system operators. In particular, the average current

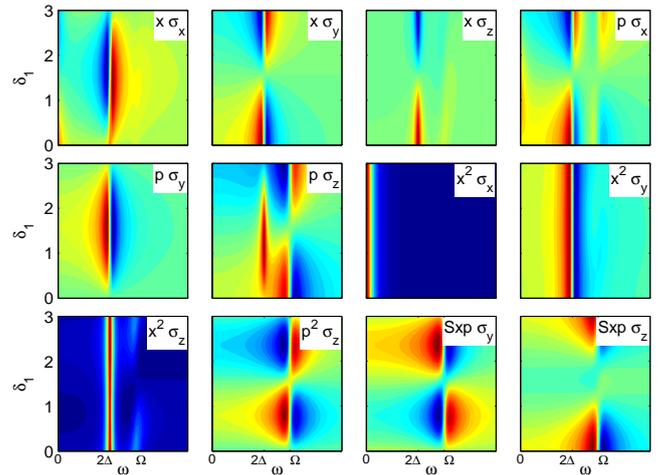


FIG. 2: (Color online) Density plot (arbitrary units) of the prefactors $S_X(\omega)$ ($X = x\sigma_x, x\sigma_y, \dots$) of the frequency-dependent noise $S(\omega) = \sum_X S_X(\omega) \langle X \rangle$ as a function of δ_1 and ω .

can be calculated using $\langle I \rangle = \frac{d}{dt} \langle \hat{n}_R \rangle = \text{Tr}_S [\hat{n}_R \dot{\rho}_S]$. The current calculated in this way reproduces Eq. (8), which was derived without using the Born-Markov approximation, as soon as $V > \Omega, 2\sqrt{\epsilon^2 + \Delta^2}$.

In order to use Eq. (10) to calculate the symmetrised noise, we make use of MacDonald’s formula, [26]

$$S(\omega) = \omega \int_0^\infty dt \sin(\omega t) \frac{d}{dt} \langle \langle \hat{n}_R^2(t) \rangle \rangle . \quad (11)$$

The calculation of the cumulant derivative $\frac{d}{dt} \langle \langle \hat{n}_R^2(t) \rangle \rangle$ using Eq. (10) leads to expressions containing cumulants $\langle \langle \hat{n}_R X \rangle \rangle$, where X is in general a product of system operators. The time evolution of these cumulants will lead to ever higher-order cumulants, such that the resulting set of differential equations is not closed. In order to make the problem tractable, several approximations are needed.

Since the qubit and the oscillator are only weakly coupled to the APC, we can assume $\gamma_1 x_0, \gamma_2 \ll \gamma_0$. In this case, we can ignore higher order cumulants of the form $\langle \langle \hat{n}_R x^2 \rangle \rangle, \langle \langle \hat{n}_R x \sigma_j \rangle \rangle$, etc. Setting these higher order cumulants to zero is often referred to as the *Gaussian* approximation.

Second, the Redfield equation leads to couplings between qubit and oscillator, e.g. $\frac{d}{dt} \langle \langle \hat{n}_R x \rangle \rangle$ contains a term depending on $\langle \langle \hat{n}_R \sigma_z \rangle \rangle$. However, since such crossterms are subleading in terms of tunnelling amplitudes, we can neglect them and thereby decouple the set of differential equations into qubit and oscillator parts.

Finally, we set $\epsilon = 0$ and $\delta_2 = 0$. This is not a crucial approximation but it simplifies the differential equations considerably.

It turns out that this set of approximations makes a solution of the Redfield equation possible and the noise

can be calculated using Eq. (11). For details, we refer to [24]. The resulting expression $S(\omega)$ contains all entries of the EVM with the exception of $\langle p^2\sigma_x \rangle$, $\langle p^2\sigma_y \rangle$ and $\langle S_{xp}\sigma_x \rangle$. The frequency-dependence is characterised by the following functions,

$$\begin{aligned}\alpha_1(\omega) &= \frac{\Gamma_1\omega^2 \sin(\delta_1) - 2V(\omega^2 - \Omega^2) \cos(\delta_1)}{\Gamma_1^2\omega^2 + (\omega^2 - \Omega^2)^2}, \\ \alpha_2(\omega) &= \frac{2V\Gamma_1\Omega \cos(\delta_1) + \Omega(\omega^2 - \Omega^2) \sin(\delta_1)}{\Gamma_1^2\omega^2 + (\omega^2 - \Omega^2)^2}, \\ \beta_1(\omega) &= \frac{V(4\Delta^2 - \omega^2)}{\Gamma_2^2\omega^2 + (\omega^2 - 4\Delta^2)^2}, \\ \beta_2(\omega) &= \frac{\Gamma_2\Delta^2}{\Gamma_2^2\omega^2 + (\omega^2 - 4\Delta^2)^2}, \\ \beta_3(\omega) &= \frac{\Gamma_2}{\Gamma_2^2 + \omega^2},\end{aligned}\quad (12)$$

which contain Lorentz and Fano shaped resonances at the characteristic frequencies of the system, $\omega = 0, \Omega, 2\Delta$. The complete expression for the noise reads $S(\omega) = \sum_X S_X(\omega) \langle X \rangle$ where X denotes all combinations of qubit and oscillator operators contained in the EVM (5). As mentioned above, it turns out that all except three of the prefactors $S_X(\omega)$ are non-vanishing and are distinguishable combinations of the functions $\alpha_{1,2}(\omega)$ and $\beta_{1,2,3}(\omega)$. Results for $S_X(\omega)$ can be found in [24]. A plot of the relevant cross-correlations' prefactors is shown in Fig. 2. Since the shapes of these functions are rather distinct, the expectation values constituting the EVM can be recovered from the total measurable noise $S(\omega)$.

Beyond the regime $V \gg \Omega, \Delta$, the Born-Markov approximation fails and we have to resort to conventional perturbation theory in order to calculate $S(\omega)$. For the case of a simple tunnelling junction, it is well-known that $S(\omega)$ has kinks at $|\omega| = |V|$ [27, 28]. In the presence of the oscillator and the qubit, we find that $S(\omega)$ shows similar features at $|\omega| = |V \pm 2\Delta|$ and $|\omega| = |V \pm \Omega|$. The amplitudes of these kinks yield additional information about the expectation values $\langle x\sigma_j \rangle$ and $\langle p\sigma_j \rangle$ ($j = x, y, z$). Details are given in [24].

Even under ideal circumstances, not all coefficients of the EVM can be determined by a current and noise measurement. However, it was shown that even incomplete knowledge of the EVM allows for the detection of entanglement for experimentally relevant states [20]. If the EVM contains a few unknown parameters a_i , we can only detect entanglement for states where Eq. (7) is violated for arbitrary a_i . Due to the special structure of the inequalities (7), this numerical problem reduces to a convex optimisation problem which can be efficiently solved using semidefinite programming as shown in [20]. A route to obtain *all* matrix elements of the EVM would be to rotate the qubit state by applying $\pi/2$ -pulses using a tunable gate voltage [5, 15, 16].

The detection of entanglement based on the EVM is highly versatile since it works even for arbitrary mixed

states. The problem is of course greatly simplified if properties of the state to be measured are known, e.g. due to a tailored preparation of the system. As an example, consider the qubit-oscillator state $|\psi\rangle = \sqrt{p_0} |\uparrow, n\rangle + i\sqrt{1-p_0} |\downarrow, n+1\rangle$, where $|n\rangle$ denotes a Fock state of the oscillator and $0 \leq p_0 \leq 1$. If the system is assumed to be in a state of this form, detection of entanglement reduces to a measurement of p_0 . For this state, it turns out that $S(\Omega, \delta_1) - S(\Omega, -\delta_1) \sim \sqrt{p_0(1-p_0)} \sin(\delta_1)$. A measurement of the noise at the oscillator resonance frequency for finite δ_1 is then sufficient to detect entanglement. Similar relations can be derived for other states. The less is known about the state, the more information has to be gained from $S(\omega, \delta_1)$. We found that even for *completely arbitrary* states, entanglement can be detected by this electronic measurement. Therefore, it is our expectation that this setup is ideally suited to detect entanglement between a continuous and a discrete quantum system.

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Supplementary Material: Detection of qubit-oscillator entanglement in nanoelectromechanical systems

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A Tunnelling phases

Phases of the tunnelling amplitudes can be created by adding a second tunnelling junction to the system and threading the resulting loop with a magnetic flux Φ . If the setup of Fig. 1 of the article is chosen, one of the tunnelling amplitudes will be x -dependent while the other one will not. A schematic drawing is shown in Supplementary Fig. 1a, and the tunnelling amplitudes of the two junctions are given by

$$\begin{aligned} t_a &= t_{a0} + t_{a1}\hat{x} + t_{a2}\sigma_z, \\ t_b &= t_{b0} + t_{b2}\sigma_z, \end{aligned} \tag{A.1}$$

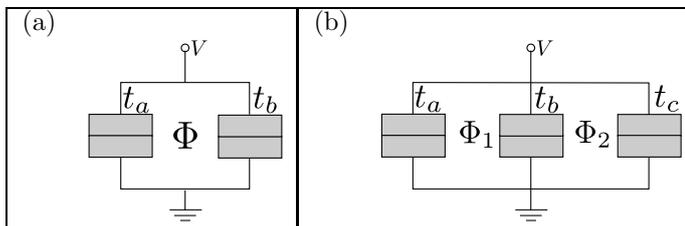
where $\hat{x} = x/x_0$. Due to the Aharonov-Bohm effect, the magnetic flux leads to a phase shift in the electronic wave function which can be absorbed in the tunnelling amplitudes. The total transmission amplitude is given by the sum of the two amplitudes,

$$t = t_a + t_b e^{-i\Phi/\Phi_0}, \tag{A.2}$$

where $\Phi_0 = h/e$ is the magnetic flux quantum. We assume that the influence of the qubit state on the amplitudes is the same for both junctions. Then, the amplitudes are related by a real constant c , $t_{a,j} = ct_{b,j}$ for $j = 0, 2$. The total tunnelling amplitude now becomes (up to an irrelevant global phase)

$$t = \sqrt{1 + c^2}(t_{a0} + t_{a2}\sigma_z) + t_{a1}e^{-i\delta_1}\hat{x}, \tag{A.3}$$

where $\delta_1 = \arg(1 + ce^{-i\Phi/\Phi_0})$. Hence, this setup provides a way to obtain a tunnelling Hamiltonian with tunable δ_1 . This has been important for the calculation of the noise. The setup can easily be extended to achieve the second tunable phase δ_2 which we used in the calculation of the current. Here, we need a third junction with an amplitude $t_c = t_{c0}$, which is decoupled from both oscillator and qubit, and two magnetic fluxes $\Phi_{1,2}$. A schematic is shown in Supplementary Fig. 1b.



Supplementary Figure 1: (a) A finite phase δ_1 can be achieved by using two junctions with amplitudes $t_a(x, \sigma_z)$ and $t_b(\sigma_z)$. The phase can be controlled by tuning the magnetic flux Φ . (b) A third junction t_c and a two magnetic fluxes $\Phi_{1,2}$ yield control over both phases $\delta_{1,2}$.

Note that in order to observe Aharonov-Bohm phases, the phase coherence length (which can be on the order of microns) must exceed the size of the setup. Moreover, the change of the area due to the fluctuating position of the

resonator is assumed to be negligible. Our analysis only covers the single-channel case. In the case of N channels in the loop, the magnitude of the Aharonov-Bohm effect is reduced by a factor $1/N$. Finally, we assume that roundtrips of electrons are ruled out by the device geometry such that electrons leave the setup after passing through either of the junctions. In this case, one can generally have $|t(\Phi)|^2 \neq |t(-\Phi)|^2$.

B Current calculation

The current operator is $I = \frac{d}{dt} \hat{n}_R = i(A\psi_L^\dagger \psi_R - A^\dagger \psi_R^\dagger \psi_L)$, where $A = \gamma Y$ and $\psi_\alpha \equiv \psi_\alpha(z=0)$. The tunnelling amplitude γ is defined in Eq. (3) of the article. The average current is in general time-dependent, and can be calculated by treating H_T as a perturbation to the non-interacting Hamiltonian $H_0 = H_{el} + H_R + H_Q$. In that case, the Kubo formula gives

$$\langle I(t) \rangle = -i \int_0^t dt' \langle \Psi(0) | [I_{\text{int}}(t), H_{T,\text{int}}(t')] | \Psi(0) \rangle, \quad (\text{B.1})$$

where the subscript ‘‘int’’ refers to the interaction picture and $|\Psi(0)\rangle$ is the state of the total system at time $t=0$. We shall assume that $|\Psi(0)\rangle$ is a product state of the fermionic ground state and the (possibly entangled) qubit-oscillator state.

The current can also be expressed in terms of the state at time t , which *to zeroth order* in the tunnelling amplitudes γ_j is $|\Psi(t)\rangle = e^{iH_0 t} |\Psi(0)\rangle$. Thus, to second order in the tunnelling amplitudes, the current becomes

$$\langle I(t) \rangle = 2 \text{Re} \int_0^t d\tau \langle \Psi(t) | [A\psi_L^\dagger \psi_R, (A^\dagger \psi_R^\dagger \psi_L)_{\text{int}}(-\tau)] | \Psi(t) \rangle. \quad (\text{B.2})$$

If the time t is large compared to the timescales of the non-interacting system $[V^{-1}, \Omega^{-1}, (\epsilon^2 + \Delta^2)^{-1/2}]$, we can extend the τ -integral to infinity. On the other hand, the above approximations are only valid for t smaller than any damping time scale, both due to the coupling to the leads ($\sim \Gamma_1^{-1}, \Gamma_2^{-1}$) and to other sources of dissipation not included in our model.

Next, we define the electronic correlation functions

$$\begin{aligned} Z_B^>(-\tau, V) &= \langle \psi_L^\dagger(0) \psi_L(-\tau) \rangle \langle \psi_R(0) \psi_R^\dagger(-\tau) \rangle \\ Z_B^<(-\tau, V) &= \langle \psi_L(-\tau) \psi_L^\dagger(0) \rangle \langle \psi_R^\dagger(-\tau) \psi_R(0) \rangle. \end{aligned} \quad (\text{B.3})$$

The averages are taken with respect to the non-interacting ground state such that these correlation functions can easily be calculated. The chemical potentials of the left and right reservoirs differ by the applied voltage V . For simplicity, we focus on the simplest case of zero temperature and a constant density of states ρ_0 . For zero voltage, one finds after Fourier transformation

$$\begin{aligned} Z_B^>(\omega) &\equiv Z_B^>(\omega, V=0) = -2\pi\rho_0^2\theta(-\omega)\omega \\ Z_B^<(\omega) &\equiv Z_B^<(\omega, V=0) = 2\pi\rho_0^2\theta(\omega)\omega. \end{aligned} \quad (\text{B.4})$$

A finite voltage leads to additional phase factors $Z_B^>(\tau, V) = e^{-iV\tau} Z_B^>(\tau)$. By exploiting $YY^\dagger = 1$, one then obtains the following expression for the average current

$$\langle I(t) \rangle = 2 \text{Re} \int_0^\infty d\tau e^{iV\tau} \left(Z_B^>(-\tau) \langle \gamma \gamma^\dagger(-\tau) \rangle - Z_B^<(-\tau) \langle \gamma^\dagger(-\tau) \gamma \rangle \right), \quad (\text{B.5})$$

where the expectation value is in the state at time t . The time dependence of the operator γ is in the interaction picture and is given by the free evolution of the oscillator and the qubit,

$$\begin{aligned} x(t) &= x \cos(\Omega t) + \frac{p}{m\Omega} \sin(\Omega t), \\ \sigma_z(t) &= \frac{\epsilon^2 + \Delta^2 \cos(2\sqrt{\epsilon^2 + \Delta^2}t)}{\epsilon^2 + \Delta^2} \sigma_z + \frac{\Delta\epsilon[1 - \cos(2\sqrt{\epsilon^2 + \Delta^2}t)]}{\epsilon^2 + \Delta^2} \sigma_x + \frac{\Delta \sin(2\sqrt{\epsilon^2 + \Delta^2}t)}{\sqrt{\epsilon^2 + \Delta^2}} \sigma_y. \end{aligned} \quad (\text{B.6})$$

The final expression (8) for the current can be obtained by using this, as well as $[x, p] = i$, $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} \sigma_k$, and the assumption that the cut-off frequency of the density of states in the leads is much larger than $V, \Omega, \sqrt{\epsilon^2 + \Delta^2}$. In conclusion, given the state $|\Psi(0)\rangle$ at $t=0$ assumed above, the expression for the average current is valid for times $V^{-1}, \Omega^{-1}, (\epsilon^2 + \Delta^2)^{-1/2} \ll t \ll \Gamma_1^{-1}, \Gamma_2^{-1}$. Note that the average current is an ensemble average, not a time average.

C Noise calculation

We now provide some details on the calculation of the symmetrised noise spectral density

$$S(\omega, t) = \frac{1}{2} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle\langle \{I(\tau), I(0)\} \rangle\rangle, \quad (\text{C.1})$$

where $\langle\langle I(\tau)I(0) \rangle\rangle = \langle I(\tau)I(0) \rangle - \langle I(\tau) \rangle \langle I(0) \rangle$ and $\{\cdot, \cdot\}$ denotes the anti-commutator. The time dependence is in the Heisenberg picture and the expectation values are taken in the state $|\Psi(t)\rangle$ at time t .

C.1 Perturbation theory

To lowest order in the tunnelling amplitudes, the noise spectral density can be approximated by

$$S(\omega, t) = \frac{1}{2} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle \Psi(t) | \{I_{\text{int}}(\tau), I(0)\} | \Psi(t) \rangle, \quad (\text{C.2})$$

where as before $I_{\text{int}}(\tau) = e^{iH_0\tau} I e^{-iH_0\tau}$. Applying the same approximations as in the calculation of the current, we find $S(\omega, t) = (\tilde{S}(\omega, t) + \tilde{S}^*(-\omega, t))/2$, where

$$\tilde{S}(\omega, t) = \int_{-\infty}^{\infty} d\tau e^{i(\omega+V)\tau} \left(Z_B^>(-\tau) \langle \gamma(\tau) \gamma^\dagger \rangle + Z_B^<(-\tau) \langle \gamma^\dagger(-\tau) \gamma \rangle \right). \quad (\text{C.3})$$

$S(\omega, t)$ is an even function of ω . For $|\omega|, \Omega, \Delta \ll |V|$, it is frequency-independent and proportional to $\text{sgn}(V) \langle I(t) \rangle$, which is due to electron shot noise. Assuming $\epsilon = 0$ and defining the function $f(\omega, V) = |\omega| + \Theta(|V| - |\omega|)(|V| - |\omega|)$, the noise spectral density becomes

$$\begin{aligned} S(\omega, t) = & \pi \rho_0^2 \\ & \times \left[2|\gamma_0| \left(|\gamma_0| + |\gamma_1| \cos \delta_1 \langle x \rangle + |\gamma_2| \cos \delta_2 \langle \sigma_z \rangle \right) f(\omega, V) - 2\Omega |\gamma_1|^2 x_0^2 + 4\Delta |\gamma_2|^2 \langle \sigma_x \rangle \right. \\ & + |\gamma_1| \left(|\gamma_0| \cos \delta_1 \langle x \rangle + |\gamma_1| \langle x^2 \rangle + |\gamma_2| \cos \delta_{21} \langle x \sigma_z \rangle \right) \left(f(\omega, V + \Omega) + f(\omega, V - \Omega) \right) \\ & + 2x_0^2 |\gamma_1| \left(|\gamma_0| \sin \delta_1 \langle p \rangle - |\gamma_2| \sin \delta_{21} \langle p \sigma_z \rangle \right) \left(f(\omega, V + \Omega) - f(\omega, V - \Omega) \right) \\ & + |\gamma_2| \left(|\gamma_0| \cos \delta_2 \langle \sigma_z \rangle + |\gamma_2| + |\gamma_1| \cos \delta_{21} \langle x \sigma_z \rangle \right) \left(f(\omega, V + 2\Delta) + f(\omega, V - 2\Delta) \right) \\ & \left. + |\gamma_2| \left(|\gamma_0| \sin \delta_2 \langle \sigma_y \rangle + |\gamma_1| \sin \delta_{21} \langle x \sigma_y \rangle \right) \left(f(\omega, V + 2\Delta) - f(\omega, V - 2\Delta) \right) \right]. \quad (\text{C.4}) \end{aligned}$$

We observe that $S(\omega, t)$ has kinks at $|\omega| = |V|$, $|\omega| = |V \pm \Omega|$ and $|\omega| = |V \pm 2\Delta|$, which can also provide information on the bipartite expectation value matrix of the oscillator-qubit system.

As determining higher-order contributions to the noise spectral density is complicated using this approach, we now turn to an alternative method.

C.2 Born-Markov approximation

In the case $V \gg \Omega, \Delta, \epsilon$, the timescale of electron dynamics will be faster than the oscillator and qubit timescales. The electron reservoirs can then be regarded as a fermionic bath $H_B = H_{el}$, coupled to the remaining system $H_S = H_R + H_Q$ via the tunnelling operator H_T . For small tunnelling, the dynamics of the reduced density matrix $\rho_S(t) = \text{Tr}_B \rho(t)$, will be determined by the Redfield equation,

$$\dot{\rho}_S(t) = -i[H_S, \rho_S(t)] - \int_0^\infty d\tau \text{Tr}_B [H_T, [H_T(-\tau), \rho_S(t) \otimes \rho_B]]. \quad (\text{C.5})$$

The time-evolution of H_T is taken with respect to the uncoupled Hamiltonian $H_S + H_B$. Expectation values of system operators can in principle be calculated by using e.g. $\frac{d}{dt} \langle x(t) \rangle = \text{Tr}_S [\dot{\rho}_S(t) x]$ and solving the resulting differential

equation. Splitting the tunnel Hamiltonian into a system and a bath part, $H_T = A\psi_L^\dagger\psi_R + \text{h.c.}$, we can rewrite this as

$$\dot{\rho}_S(t) = -i[H_S, \rho_S(t)] - \int_0^\infty d\tau \{Z_B^>(-\tau, V)[A, A^\dagger(-\tau)\rho_S(t)] - Z_B^<(-\tau, V)[A, \rho_S(t)A^\dagger(-\tau)] + \text{h.c.}\}, \quad (\text{C.6})$$

where we used the bath correlation functions defined in Eq. (B.3) and the free evolution of the qubit-oscillator operators is given in Eq. (B.6). Therefore, $A(t)$ contains terms oscillating at the frequencies Ω and $2\sqrt{\epsilon^2 + \Delta^2}$. The τ -integration in Eq. (C.6) can then be performed and its result can be absorbed into generalised transmission probabilities Γ_{KJ}^{\pm} where K and J run over the system operators, $K, J \in \{x, p, \sigma_x, \sigma_y, \sigma_z\}$. The average current is defined as $\langle I(t) \rangle = \text{Tr}_S[\dot{\rho}_S(t)n_R]$. Using the Redfield equation, it can be written as

$$\begin{aligned} \langle I \rangle &= - \sum_{KJ} \left\{ \Gamma_{KJ}^{\rightarrow} \langle [n_R, YK]Y^\dagger J \rangle - \Gamma_{KJ}^{\leftarrow} \langle Y^\dagger J[n_R, YK] \rangle + \text{c.c.} \right\} \\ &= \sum_{KJ} \left\{ \Gamma_{KJ}^{\rightarrow} \langle KJ \rangle - \Gamma_{KJ}^{\leftarrow} \langle JK \rangle + \text{c.c.} \right\}, \end{aligned} \quad (\text{C.7})$$

where we evaluated the expression using $[Y, n_R] = Y$ and $YY^\dagger = Y^\dagger Y = \mathbb{1}$ while all other commutators of n_R and Y with the system operators K, J vanish. The calculation of the transmission coefficients is a straightforward task and the average current for $V > \Omega$, $\sqrt{\epsilon^2 + \Delta^2}$ coincides exactly with the result from the perturbation theory.

In the following, we consider the case $\epsilon = \delta_2 = 0$. The symmetrised frequency-dependent current noise is defined by Eq. (C.1). In order to transform this into an expression which can be calculated using the Redfield equation, we use MacDonald's formula, see Eq. (11) of the article. The time-derivative of the cumulant of the number of transferred electrons $\langle\langle n_R^2 \rangle\rangle = \langle n_R^2 \rangle - \langle n_R \rangle^2$ can be calculated using Eq. (C.6). The noise turns out to have a zero-frequency component proportional to the average current $S(\omega = 0) = 2e \langle I \rangle$. After subtracting this term, one finds

$$\begin{aligned} \frac{d}{dt} \langle\langle n_R^2 \rangle\rangle - \langle I \rangle &= \frac{d}{dt} \langle n_R^2 \rangle - 2 \langle n_R \rangle \langle I \rangle - \langle I \rangle \\ &= 4V \sqrt{\frac{\Gamma_0 \Gamma_1}{\Omega}} \cos(\delta_1) \langle\langle n_R \hat{x} \rangle\rangle + 2\sqrt{\Gamma_0 \Gamma_1 \Omega} \sin(\delta_1) \langle\langle n_R \hat{p} \rangle\rangle + \frac{V\Gamma_1}{\Omega} \langle\langle n_R \hat{x}^2 \rangle\rangle \\ &\quad + \frac{2\Delta\Gamma_2}{V} \langle\langle n_R \sigma_x \rangle\rangle + 4\sqrt{V\Gamma_0 \Gamma_2} \langle\langle n_R \sigma_z \rangle\rangle \\ &\quad + 2\sqrt{\frac{V\Gamma_1 \Gamma_2}{\Omega}} \cos(\delta_1) \langle\langle n_R \hat{x} \sigma_z \rangle\rangle - 2\Delta \sqrt{\frac{\Gamma_1 \Gamma_2}{V\Omega}} \sin(\delta_1) \langle\langle n_R \hat{x} \sigma_y \rangle\rangle + \sqrt{\frac{\Gamma_1 \Gamma_2 \Omega}{V}} \sin(\delta_1) \langle\langle n_R \hat{p} \sigma_z \rangle\rangle, \end{aligned} \quad (\text{C.8})$$

where the oscillator coordinate was measured in units of its zero-point motion, $\hat{x} = x/x_0$, $\hat{p} = 2x_0 p$ where $x_0 = 1/\sqrt{2m\Omega}$. Next, we assume that the bare tunnelling amplitude is large compared to the coupling to the qubit and the oscillator, ie. $\gamma_0 \gg \gamma_1 x_0, \gamma_2$. Since all of the above cumulants vanish in the absence of coupling to the qubit and the oscillator, they have to be on the order of these couplings γ_1, γ_2 . Therefore, cumulants containing quadratic operators in addition to n_R are subleading and we can use

$$\frac{d}{dt} \langle\langle n_R^2 \rangle\rangle - \langle I \rangle = 4V \sqrt{\frac{\Gamma_0 \Gamma_1}{\Omega}} \cos(\delta_1) \langle\langle n_R \hat{x} \rangle\rangle + 2\sqrt{\Gamma_0 \Gamma_1 \Omega} \sin(\delta_1) \langle\langle n_R \hat{p} \rangle\rangle + \frac{2\Delta\Gamma_2}{V} \langle\langle n_R \sigma_x \rangle\rangle + 4\sqrt{V\Gamma_0 \Gamma_2} \langle\langle n_R \sigma_z \rangle\rangle. \quad (\text{C.9})$$

Since for a Gaussian distribution, all cumulants beyond quadratic order vanish, we call this the Gaussian approximation. The next step is to calculate the time-dependence of the cumulants which can again be accomplished using the Redfield equation (C.6). In order to simplify the expressions, we can assume $\langle x \rangle = \langle p \rangle = \langle \sigma_z \rangle = 0$ since finite values for these expectation values would merely lead to a renormalised tunnelling amplitude. As an example, consider the differential equation for $\langle\langle n_R p \rangle\rangle$,

$$\frac{d}{dt} \langle\langle n_R p \rangle\rangle = (\dots) - 2\Delta \sqrt{\frac{\Gamma_1 \Gamma_2}{V\Omega}} \cos(\delta_1) \langle\langle n_R \sigma_y \rangle\rangle - \sqrt{\frac{V\Gamma_1 \Gamma_2}{\Omega}} \sin(\delta_1) \langle\langle n_R \sigma_z \rangle\rangle - \Omega \langle\langle n_R x \rangle\rangle - \Gamma_1 \langle\langle n_R p \rangle\rangle, \quad (\text{C.10})$$

where (\dots) denotes a number of n_R -independent terms and the initial condition reads $\langle\langle n_R p \rangle\rangle_{t=0} = 0$. For the reason mentioned above, the terms containing cumulants of qubit operators are subleading and can be dropped. The same is

true for the remaining differential equation, such that equations describing qubit cumulants and oscillator cumulants decouple. This makes an analytical solution of the problem possible. The result is an expression for the noise of the form $S(\omega) = \sum_X S_X(\omega) \langle X \rangle$, where X denotes qubit or oscillator operators or products of these. The prefactors $S_X(\omega)$ for the cross-terms turn out to be given by

$$\begin{aligned}
S_{x\sigma_x} &= 2\sqrt{2M\Gamma_0\Gamma_1\Gamma_2}(\Delta/V) [\Omega\alpha_2(\omega) + 2V\beta_3(\omega) \cos(\delta_1) + 8V\beta_2(\omega) \cos(\delta_1) - 4V\beta_1(\omega) \sin(\delta_1)] \\
S_{x\sigma_y} &= 2\sqrt{2M\Gamma_1\Gamma_2/V\Delta} [8V\Gamma_0\beta_1(\omega) \cos(\delta_1) + \Gamma_2\beta_3(\omega) \sin(\delta_1)] \\
S_{x\sigma_z} &= 2\sqrt{2MVT\Gamma_1\Gamma_2} [-2\Gamma_2(\Delta/V)^2\beta_3(\omega) \cos(\delta_1) + 16V\Gamma_0\beta_2(\omega) \cos(\delta_1) + 2\Gamma_0\Omega\alpha_2(\omega)] \\
S_{p\sigma_x} &= 2\sqrt{2\Gamma_0\Gamma_1/M\Gamma_2}(\Delta/V) [\alpha_1(\omega) + 2\beta_1(\omega) \cos(\delta_1) + \beta_3(\omega) \sin(\delta_1)] \\
S_{p\sigma_y} &= -\sqrt{2\Gamma_1\Gamma_2/(VM)}(\Delta/V) [\Gamma_2\beta_3(\omega) \cos(\delta_1) - 8V\Gamma_0\beta_1(\omega) \sin(\delta_1)] \\
S_{p\sigma_z} &= 4\sqrt{2V\Gamma_1\Gamma_2/M\Gamma_0} [\alpha_1(\omega) + 4\beta_2(\omega) \sin(\delta_1)] \\
S_{x^2\sigma_x} &= 2M\Gamma_1\Gamma_2\Delta\beta_3(\omega) \\
S_{x^2\sigma_y} &= 4M\sqrt{\Gamma_0\Gamma_2/V\Gamma_1}\Delta [-\Omega\alpha_2(\omega) \sin(\delta_1) + 2V\beta_1(\omega)] \\
S_{x^2\sigma_z} &= 4M\Gamma_1\sqrt{V\Gamma_0\Gamma_2} [4V\beta_2(\omega) + \Omega\alpha_2(\omega) \cos(\delta_1)] \\
S_{p^2\sigma_x} &= 0 \\
S_{p^2\sigma_y} &= 0 \\
S_{p^2\sigma_z} &= 2\sqrt{\Gamma_0\Gamma_2/V}(\Gamma_1/M)\alpha_1(\omega) \sin(\delta_1) \\
S_{S_{xp}\sigma_x} &= 0 \\
S_{S_{xp}\sigma_y} &= -2\sqrt{\Gamma_0\Gamma_2/V\Gamma_1}\Delta\alpha_1(\omega) \sin(\delta_1) \\
S_{S_{xp}\sigma_z} &= \sqrt{\Gamma_0\Gamma_2/V\Gamma_1} [2V\alpha_1(\omega) \cos(\delta_1) + \Omega\alpha_2(\omega) \sin(\delta_1)], \tag{C.11}
\end{aligned}$$

where the functions $\alpha_{1,2}(\omega)$ and $\beta_{1,2,3}(\omega)$ were defined in Eq. (12) of the article.