Quantum state transfer in arrays of flux qubits

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Abstract. In this work, we describe a possible experimental realization of Bose's idea to use spin chains for short distance quantum communication [S. Bose, *Phys. Rev. Lett.* **91** 207901]. Josephson arrays have been proposed and analyzed as transmission channels for systems of superconducting charge qubits. Here, we consider a chain of persistent current qubits, that is appropriate for state transfer with high fidelity in systems containing flux qubits. We calculate the fidelity of state transfer for this system. In general, the Hamiltonian of this system is not of XXZ-type, and we analyze the magnitude and the effect of the terms that do not conserve the z-component of the total spin.

1. Introduction

Recently, the idea to use quantum spin chains for short-distance quantum communication was put forward by Bose [1]. He showed that an array of spins (or spin-like two level systems) with isotropic Heisenberg interaction is suitable for quantum state transfer. The advantage of spin chains as transmission lines is the fact, that they do not need to have controllable couplings between the qubits or complicated gating schemes to achieve high transfer fidelity. An initial state is prepared at one end of the chain at time $t = 0$, and after a certain time t_1 is measured at the other. The fidelity of quantum communication averaged over all pure input states on the Bloch sphere is taken as a measure of the transmission quality.

Bose showed that for short chains (number of spins $\simeq 100$) the average fidelity is quite high, greater than $2/3$, which is the highest fidelity of transmission through a classical channel [2]. In a homogeneous chain, i.e. if all coupling constants are the same, the information about the input state is dispersed between the spins at all times $t > 0$. Therefore the fidelity is always less then unity.

Some methods were proposed to achieve perfect state transfer with fidelity one. A special form of the Hamiltonian with spatially varying coupling constants between the qubits allows to avoid dispersion [3, 4]. Another method is to form Gaussian wave packets (with low dispersion) by encoding the information using multiple spins [5]. Also the combination of two spin chains [6] can be used to achieve perfect state transfer. This method has the advantage that it can be implemented using almost any two spin chains and is stable to fluctuations of the chain parameters [7]. However, the time after which perfect state transfer is achieved grows if the individual fidelities of the chains decrease. It is therefore advantageous to have single chains with high fidelity to implement this improved method.

Quantum state transfer can be implemented using any type of two-level systems. However, it is preferable to use a technology that is adapted to the quantum information hardware that is supposed to be coupled by the transmission line. One of the most promising architectures of quantum computing devices are superconducting circuits, for example charge, flux and charge-flux qubits. In recent years these were intensively studied both theoretically and experimentally.

A possible realization of an effective transmission line for charge qubits was described in [8]. There, the fidelity of state transfer through Josephson junction arrays and the influence of static disorder and dynamical noise was analyzed.

2. Arrays of persistent-current qubits

In this work we consider a line of persistent-current qubits [9]. We will show that it is appropriate for state transfer with high fidelity in systems containing flux qubits. A persistent current qubit [10] is a superconducting loop with three Josephson junctions, see Fig. 1. We assume that the left and right Josephson junctions have capacitance

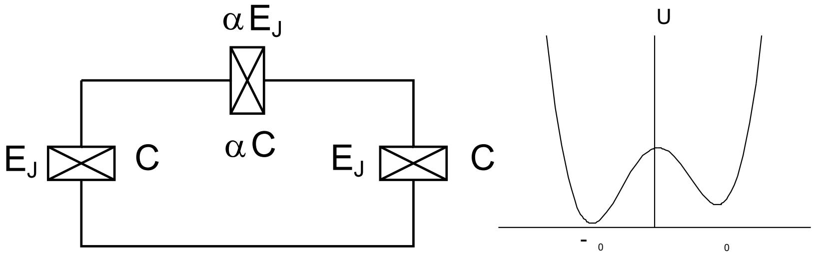

Figure 1. Persistent-current qubit and Josephson energy as a function of the relative phase of left and right Josephson junctions.

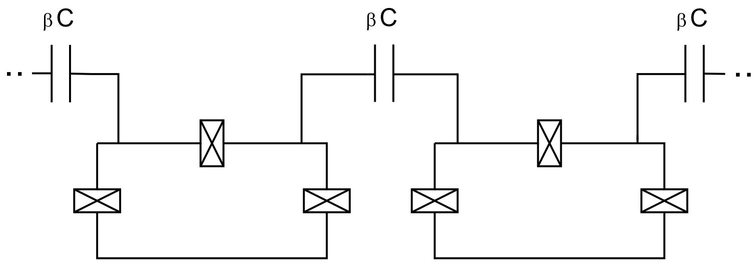

Figure 2. Capacitively coupled qubits.

C and Josephson energy E_J, the central junction is characterized by a capacitance αC and Josephson energy αE_J with $\alpha < 1$. The gate capacitances (not shown in the figure) are equal γC. The Hamiltonian of the qubit

$$H_0 = -\Delta_0 \sigma^x - B\sigma^z \tag{1}$$

is the same as that of a spin-$\frac{1}{2}$ particle in a magnetic field. The eigenstates $|0\rangle \equiv |\downarrow\rangle$ and $|1\rangle \equiv |\uparrow\rangle$ of σ^z correspond to clockwise and counterclockwise currents. The coefficient Δ_0 is a tunneling amplitude between these states and B depends on the flux through the qubit Φ and the modulus of the circulating current I_p

$$B = I_p(\Phi)\left(\Phi - \frac{1}{2}\Phi_0\right) , \tag{2}$$

here, $\Phi_0 = h/(2e)$ is the flux quantum. The circulating current I_p depends on the magnetic frustration, i.e. the amount of external magnetic flux in the loop in units of the flux quantum [11]. The effective magnetic field B is determined by the qubit parameters and the external magnetic flux.

We assume, that the temperature is low enough, i.e. $k_B T$ is smaller than the energy of the state $|1\rangle$, so we can neglect thermal fluctuations.

Persistent-current qubits can be capacitively coupled (with coupling capacitance βC, see Fig. 2) to form a one-dimensional array, that for $\beta \gg 1$ has the Hamiltonian

$$H = -\sum_{i=2}^{N}[J_{xy}(\sigma_i^+\sigma_{i-1}^- + \sigma_i^-\sigma_{i-1}^+) + J_z\sigma_i^z\sigma_{i-1}^z] - \sum_{i=1}^{N}(\Delta\sigma_i^x + B\sigma_i^z) . \tag{3}$$

The terms $J_z \sigma_i^z \sigma_{i+1}^z$ are due to the small inductive coupling between adjacent qubits. Here $J_z = 2M_{q,q} I_p^2$, where $M_{q,q}$ is their mutual magnetic inductance. The coupling constant J_z could in principle be increased by a common Josephson junction between two neighboring qubits [9]. The tunneling amplitude Δ between the states $|0\rangle$ and $|1\rangle$ of the coupled qubits differs from the value Δ_0 for individual non-coupled qubits, because coupling suppresses independent tunneling events in which only one qubit changes its state. Also, simultaneous tunneling events $|11\rangle \longleftrightarrow |00\rangle$ for two neighboring qubits are suppressed and therefore we neglect such processes in our model. Correlated tunneling events $|10\rangle \longleftrightarrow |01\rangle$ are unaffected by the coupling.

The Hamiltonian (3) contains a term $\Delta \sum_i \sigma_i^x$, i.e. it does not conserve the z-component of the total spin (which is equivalent to the number of sites in the excited state $|1\rangle$). Therefore, the theory proposed in [1] is not valid in our case. However, if $\beta \gg 1$ Δ is much less than J_{xy} [9] and we can neglect this term at first. Later we will use perturbation theory to analyze how nonzero values of Δ affect the results.

We assume that the gate capacitances are equal to γC. As was shown in [11] using the quasiclassical approach, Δ_0 can be obtained as

$$\Delta_0 = \sqrt{E_J E_C} \sqrt{\frac{2(4\alpha^2 - 1)}{\alpha(1+\gamma)}} \exp(-4\sqrt{M\alpha E_J}(\sqrt{1-\alpha^2/4} - \frac{\arccos(\alpha/2)}{2\alpha})) \quad (4)$$

where

$$M = \frac{\hbar^2}{E_C} \frac{1 + 2\alpha + \gamma}{4} \ . \quad (5)$$

We now want to consider two interacting qubits that are coupled by a capacitor βC, see Fig. 2. The dynamics of the qubit can be described by the motion of a fictitious particle in a potential (see Fig. 1) with two local minima, that correspond to the states $|0\rangle$ and $|1\rangle$ [10]. Therefore the collective dynamics of the two qubits can be described by the motion of a particle in a two-dimensional potential with four minima. Without interaction the effective mass of the particle is M, see Eq. (5). When two qubits are connected by the capacitor, the effective mass to move in $(0,0) \leftrightarrow (1,1)$ direction is $M + 2m^*$ (here $m^* = (\hbar/2e)^2 \beta C$), the effective mass for independent qubit tunneling events is $M + m^*$ and the effective mass for tunneling in $(1,0) \leftrightarrow (0,1)$ direction is equal to M. From these formulas one can see that the tunneling is suppressed in all directions except $(1,0) \leftrightarrow (0,1)$, if $m^* \gg M$. Due to this fact state transfer with high fidelity is possible.

Using the WKB-approach and realistic qubit parameters from [11] and [10], namely $\alpha = 0.75$, $\gamma = 0.02$, we calculate Δ and J_{xy} for our Hamiltonian:

$$\Delta = \Delta_0 \exp(-0.49\sqrt{E_J/E_C}(\sqrt{1+\beta/5} - 1)) \ , \quad (6)$$

$$4J_{xy} = \Delta_0 e^{-0.49\sqrt{E_J/E_C}} (1 - e^{-0.98\sqrt{E_J/E_C}(\sqrt{1+\beta/5}-1)}) \ . \quad (7)$$

With $E_J/E_C \approx 100$, we obtain

$$\Delta/\Delta_0 = \exp(-4.9(\sqrt{1+\beta/5} - 1)) \ . \quad (8)$$

Therefore, independent tunneling is effectively suppressed for $\beta \sim 10$. Δ and $4J_{xy}$ coincide for $\beta = 15$. For $\beta = 20$, $4J_{xy}$ is three times larger and for $\beta = 30$ it is 25 times larger than Δ. In this case, as we will show later, Δ can be neglected.

For $\Delta = 0$, the Hamiltonian (3) is that of an asymmetric (XXZ) Heisenberg model in the presence of a magnetic field,

$$H_L = -\sum_{i=2}^{N}[J_{xy}(\sigma_i^+\sigma_{i-1}^- + \sigma_i^-\sigma_{i-1}^+) + J_z\sigma_i^z\sigma_{i-1}^z] - \sum_{i=1}^{N} B\sigma_i^z \,. \tag{9}$$

We now want to calculate the fidelity of the state transfer. The chain is initialized in the state $|00...00\rangle$ by first choosing a large negative value for the parameter B, see Eqs. (9) and (2). Then, the first qubit is prepared in the state $|\psi_{in}\rangle$ i.e., the total state of the array is $|\psi_{in}, 00...00\rangle$. This is not an eigenstate of the Hamiltonian (9), therefore the system will evolve in time. After a time t the state of the last qubit is read out. In general the last qubit will be in a mixed state, which is described by a density matrix ρ_{out}. Following [1], we average the fidelity over all pure input states on the Bloch sphere

$$F(t) = \frac{1}{4\pi}\int \langle\psi_{in}|\rho_{out}(t)|\psi_{in}\rangle \mathrm{d}\Omega \tag{10}$$

to obtain a quantity $1/2 \leq F(t) \leq 1$ that measures the quality of transmission independent of $|\psi_{in}\rangle$.

3. Calculation of the average fidelity

We perform our calculations in the basis $|k\rangle = |00...010...0\rangle$ for which the spin in the k-th qubit is in the state $|1\rangle$ and all others are in the state $|0\rangle$. The Hamiltonian (9) of the array commutes with the z-component of the total spin $\sum_i \sigma_i^z$. Therefore we can use the results of [1] to calculate the average fidelity in terms of $f_{1,N}^N(t) = \langle 1|e^{-iH_Lt}|N\rangle$ i.e., the transition amplitude of the excitation over the array. The average fidelity can then be expressed as

$$F(t) = \frac{1}{2} + \frac{|f_{1,N}^N(t)|^2}{6} + \frac{|f_{1,N}^N(t)|\cos(\gamma)}{3} \,, \tag{11}$$

where $\gamma = \mathrm{Arg}(f_{1,N}^N(t))$ is the argument of the complex quantity $f_{1,N}^N(t)$.

Varying the magnetic field one can make γ a multiple of 2π to maximize the average fidelity, such that the maximum fidelity will correspond to the maximum of $|f_{1,N}^N(t)|$. Furthermore, the fidelity of any state transfer is unity, if the modulus of the amplitude to transmit the state $|1\rangle$ across the array is unity. The fidelity for a given state $|\psi_{in}\rangle = \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle$ in the case $f_{1,N}^N(t) \equiv f = |f|$ is

$$F(\theta,\varphi) = \frac{1+f}{2} + \cos(\theta)\frac{1-f^2}{2} + \cos^2(\theta)\frac{f^2-f}{2} \,. \tag{12}$$

It changes monotonically from 1 for the $|0\rangle$ state to f^2 for the $|1\rangle$ state. For $f \neq |f|$ the fidelity can have a local minimum for $\theta = \arccos(\frac{1-|f|^2}{2(|f|\cos(\gamma)-|f^2|)})$ if $|f| > \cos(\gamma)/3 + \sqrt{\cos^2(\gamma)+3}/3$.

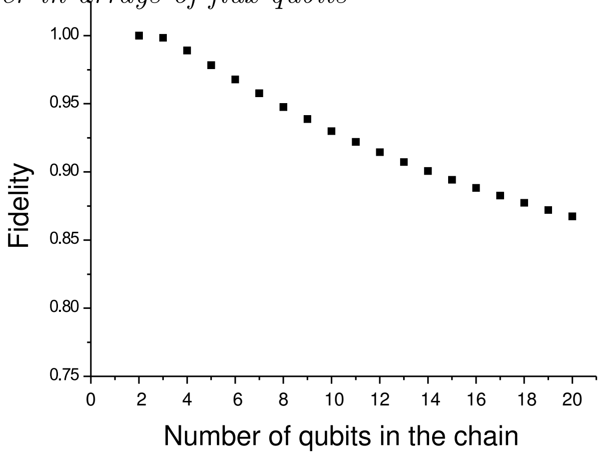

Figure 3. First fidelity maximum for an array with $\alpha = 0.75$, $\gamma = 0.02$, $E_J/E_C = 100$, $\beta = 30$ and $E_J = 3$GHz, $a = 0.1$.

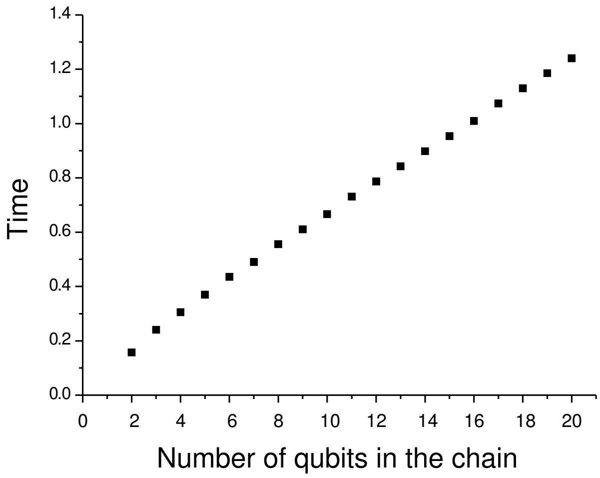

Figure 4. Time (in units of $1/E_J$) at which the first fidelity maximum is achieved. It is proportional to the length of the chain and depends on the coefficient J_{xy}.

We will now calculate $|f_{1,N}^N(t)|$ in the case $\Delta = 0$. The eigenfunctions of H_L can be described as follows:

$$|\tilde{k}\rangle = \sum_{n=1}^{N} b_{k,n}|n\rangle \ . \tag{13}$$

From the Schrödinger equation

$$H_L|\tilde{k}\rangle = (B(N-2) - J_z(N-5))|\tilde{k}\rangle - 2J_z(b_{k,1}|1\rangle + b_{k,N}|N\rangle)$$
$$-4J_{xy}(b_{k,2}|1\rangle + \sum_{n=2}^{N-1}(b_{k,n-1} + b_{k,n+1})|n\rangle + b_{k,N-1}|N\rangle) \ , \tag{14}$$

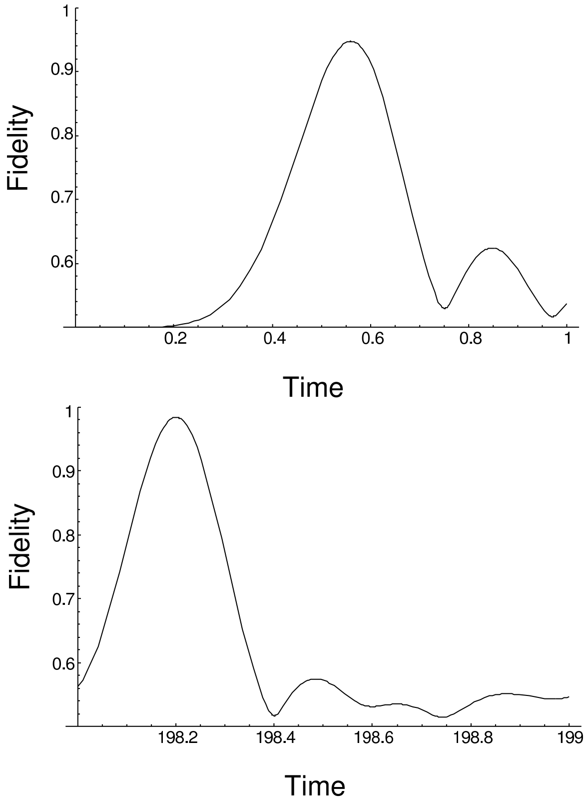

Figure 5. Fidelity as a function of time (in units of $1/E_J$) for a chain with $N = 8$. Upper panel: first fidelity maximum at small times. Lower panel: fidelity maxima around $t = 198$. The parameters are chosen as in Fig. 3.

we obtain the following system of equations for the coefficients $b_{k,n}$

$$\begin{cases} b_{k,n-1} + b_{k,n+1} = Db_{k,n} \quad (n \in [2, N-1]) \\ ab_{k,1} + b_{k,2} = Db_{k,1} \\ ab_{k,N} + b_{k,N-1} = Db_{k,N} \end{cases} \tag{15}$$

where $a = J_z/2J_{xy}$ and D is a constant. From the first two equations $b_{k,i}$ can be expressed in terms of $b_{k,1}$ as

$$b_{k,i} = P_i(D_k)b_{k,1} \ , \tag{16}$$

here $D_k, k = 1, ..., N$ are the roots of

$$(D - a)P_N(D) = P_{N-1}(D) \ . \tag{17}$$

$P_i(D)$ is a polynomial, that is determined recursively

$$P_1 = 1, \quad P_2 = D - a, \quad P_i = DP_{i-1} - P_{i-2}, \quad i = 3, ..., N \ . \tag{18}$$

The coefficient $b_{k,1}$ can be found from the normalization conditions

$$\langle \tilde{k} | \tilde{m} \rangle = \delta_{k,m} \Rightarrow b_{k,1}^2 = \frac{1}{P_1^2(D_k) + ... + P_N^2(D_k)} \ . \tag{19}$$

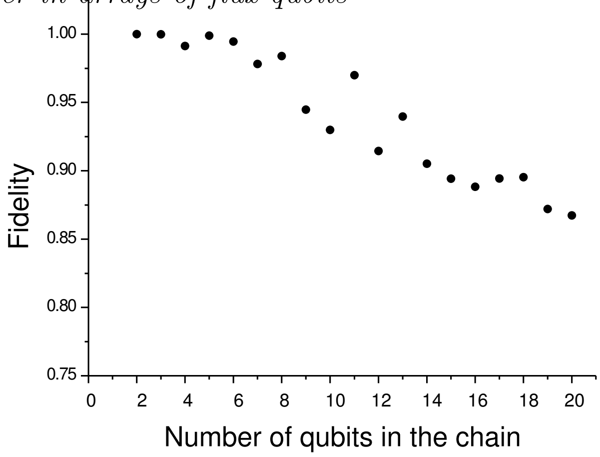

Figure 6. Fidelity maxima for times less than $4000/J_z$, all the chain parameters are as in Fig. 3.

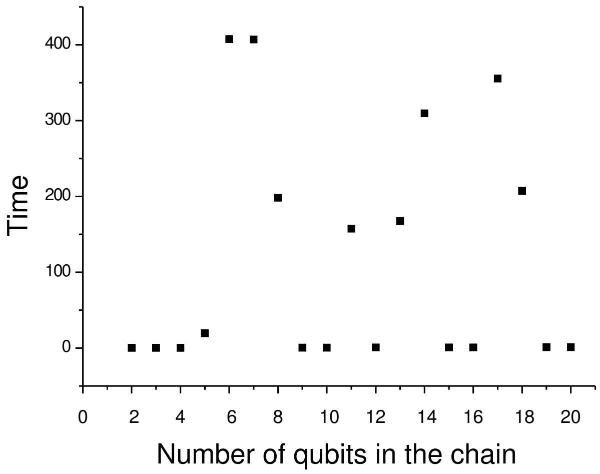

Figure 7. Times (in units of $1/E_J$) at which the fidelity maxima in Fig. 6 are achieved.

Thus we have determined the eigenfunctions of the Hamiltonian and can find its eigenenergies

$$E_k = -J_z(N-5) + B(N-2) - 4D_k J_{xy} \ . \tag{20}$$

Setting $E_0 = 0$, we obtain

$$E_k = 2B + 4J_z - 4D_k J_{xy} \ . \tag{21}$$

The transition amplitude of the excitation through the array is given by

$$f_{1,N}^N(t) = \sum_{k=1}^{N} \langle \tilde{k}|1\rangle\langle N|\tilde{k}\rangle e^{-iE_k t} = \sum_{k=1}^{N} b_{k,1} b_{k,N} e^{-iE_k t} \ . \tag{22}$$

Using these formulas we have numerically calculated the average fidelities for different chain lengths and ratios $a = J_z/2J_{xy}$. The most relevant quantities for practical purposes are the first fidelity maxima, see Fig. 3 and Fig. 4, that we will call "fidelity" in the rest of this article.

For short-length chains the average fidelity is higher than 0.9. This makes persistent qubit arrays good candidates for transmission lines in quantum computers, that are based on flux degrees of freedom. Also they can be effectively used in the two-chain method proposed for achieving perfect state transfer [6]. The fidelity has a complicated oscillating behavior as a function of time, see Fig. 5. There are many local maxima, and the first of them is usually not the global maximum. Therefore, waiting long enough, we can achieve a higher fidelity. This can be seen by comparing Fig. 3 and Fig. 6.

However, the waiting times, i.e. the times, at which the maximum peaks of the fidelity shown in Fig. 6 occur are much longer than for the first maximum. Therefore, from a practical point of view the first maxima in the fidelity are more relevant.

Decoherence is another important reason why practical realizations of our proposal would have to focus on the first fidelity maximum. Like any physical realization of a qubit, flux qubits are characterized by a finite dephasing time, and in a recent experiment times of order $\tau_\phi \approx 20$ns were reported for a *single* flux qubit [12]. Since the time for the appearance of the first fidelity maximum is of order $\hbar L/E_J$. As a simple estimate of the effects of decoherence, we compare this time with the dephasing time, which leads to a a limit of the length of the array of $L \sim \tau_\phi E_J/\hbar \sim 100$. Additional maxima after the first one will be further reduced by decoherence since they correspond to states traversing the array more than once.

To maximize the fidelity $\gamma = \mathrm{Arg}(f_{1,N}^N(t))$ has to be chosen equal to zero. This can be done by varying the magnetic field, so that $-2Bt + \gamma_0 = 2\pi n$. Here γ_0 is transition amplitude phase for $B = 0$. To achieve more control of the qubit parameters the central junction can be replaced by a SQUID [11].

The works of Bose [1] and Christandl *et al.* [3] correspond to spin chains with a particular form of the Hamiltonian H_L ($J_z/2J_{xy} = 1$, $J_z/2J_{xy} = 0$). We have checked that in these limits our results agree with [1] and [3].

As mentioned above, the Hamiltonian of the real chain contains a term $\Delta \sum_{i=1}^N \sigma_i^x$, that does not conserve the z-component of the total spin (i.e. the number of excitations). Δ is small, however, i.e. we can use perturbation theory to analyze the influence of this term on the average fidelity. In this case we need to do calculations in a larger (2^N+1)-dimensional space, because in principle any number of excitations is possible. One can easily show, that in zero-order approximation the fidelity and the $N+1$ lowest eigenstates will be the same as in the unperturbed case. The first-order corrections are zero, because $\langle k|\sigma_i^x|k\rangle = 0$ for the lowest eigenstates. So only the second-order terms, which are proportional to Δ^2, affect the fidelity. The influence of the symmetry-breaking term therefore vanishes quadratically with Δ.

From Fig. 8 one can see that for qubits with the parameters mentioned in Fig. 3 it is sufficient to choose coupling capacitors with about 25-30 times the junction capacitance. In this case we can neglect the influence of Δ. For long times this term becomes more important. This is another reason why only the first maxima are useful for practical

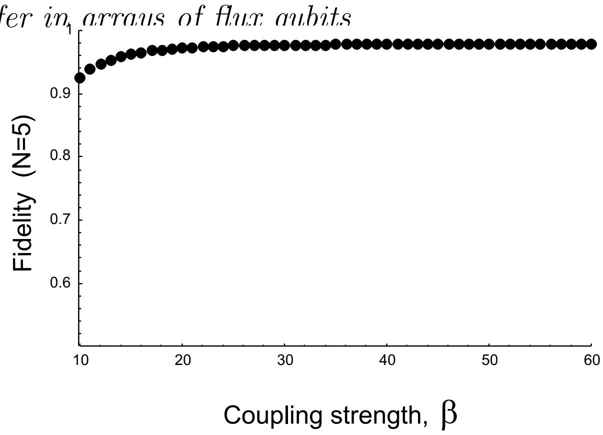

Figure 8. Fidelity dependence on β for chain length $N = 5$.

realizations of high-fidelity transmission lines. One can, in principle, raise β to make the Δ-term less important for the maxima that occur later, but in this case the charging energy will increase and this will influence the fidelity and the properties of the qubit.

4. Conclusions

We have shown that a persistent-current qubit array is a good candidate for quantum state transfer with high fidelity in flux-qubit based quantum computers. For short-length chains the average fidelity of state transfer is higher than 0.9. Therefore, this type of array can be effectively used in the two-chain algorithm [6] for achieving perfect state transfer. The influence of the term proportional to $\Delta\sigma^x$, that does not commute with the z-component of the total spin, is quadratic in Δ and can be neglected at small times for $\beta \gg 1$.

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References

[1] Bose S 2003 *Phys. Rev. Lett.* **91** 207901
[2] Horodecki M, Horodecki P and Horodecki R 1999 *Phys. Rev.* A **60** 1888
[3] Christandl M, Datta N, Ekert A and Landahl A J 2004 *Phys. Rev. Lett.* **92** 187902
[4] Paternostro M, Palma G M, Kim M S and Falci G 2004 *Phys. Rev.* A **71** 042311
[5] Osborne T J and Linden N 2004 *Phys. Rev.* A **69** 052315
[6] Burgarth D and Bose S 2005 *Phys. Rev.* A **71** 052315
[7] Burgarth D and Bose S 2005 *New J. Phys.* **7** 135
[8] Romito A, Fazio R and Bruder C 2005 *Phys. Rev.* B **71** 100501
[9] Levitov L S, Orlando T P, Majer J B and Mooij J E 2001 Quantum spin chains and Majorana states in arrays of coupled qubits *Preprint* cond-mat/0108266
[10] Mooij J E, Orlando T P, Levitov L S, Tian L, van der Wal C H and Lloyd S 1999 *Science* **285** 1036
[11] Orlando T P, Mooij J E, Tian L, van der Wal C H, Levitov L S, Lloyd S and Mazo J J 1999 *Phys. Rev.* B **60** 15398
[12] Chiorescu I, Nakamura Y, Harmans C J P M and Mooij J E 2003 *Science* **299**, 1869