Variable electrostatic transformer: controllable coupling of two charge qubits

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(July 25, 2003)

We propose and investigate a novel method for the controlled coupling of two Josephson charge qubits by means of a variable electrostatic transformer. The value of the coupling capacitance is given by the discretized curvature of the lowest energy band of a Josephson junction, which can be positive, negative, or zero. We calculate the charging diagram of the two-qubit system that reflects the transition from positive to negative through vanishing coupling. We also discuss how to implement a phase gate making use of the controllable coupling.

Following experimental demonstrations of individual qubits implemented with Josephson junctions operated in the charge [1,2] or flux [3–6] regimes, a lot of interest is now focused on building multi-qubit Josephson circuits. Recently, there has been an encouraging first experimental step [7] demonstrating quantum-coherent dynamics of two charge qubits coupled directly through a capacitance. The coupling strength could not be varied in this experiment, since there is no simple way of changing the electrostatic capacitance between two metallic islands. A controlled coupling of charge qubits would be highly desirable. Many two-qubit transformations that could be implemented with a controllable coupling become more difficult or impossible at constant coupling, since in the presence of interactions one needs to correct for different dynamic phases of the two-qubit states. Two-qubit gates with untunable couplings have been suggested [8], however, the gate design becomes significantly more complex.

For flux qubits, the problem of a controlled coupling can be solved with variable flux transformers (see, e.g., [9]) which can be implemented naturally with the standard tools of superconductor electronics [10]. For other types of "magnetic" qubits, e.g., spin qubits, the possibility to control the qubit coupling both in sign and absolute value has been predicted theoretically [11] (for recent experimental progress, see [12]). For charge qubits, however, a comparably natural solution has been lacking although the importance of the interaction control problem has been recognized for some time. Suggested controlled coupling methods include direct spatial separation of qubit states by adiabatic transfer along junction arrays [13] or coupling through superconducting resonators [14–20]. The first one is too complicated to be practical at present. The second requires relatively large inductances of small geometric dimension that are fundamentally difficult to produce, in particular without creating additional dissipation at the relevant large frequencies.

The purpose of this work is to analyze a new circuit to implement a controllable coupling of charge qubits. The circuit, shown in Fig. 1, generalizes the simple capacitive coupling used in [7] and makes use of the following basic feature of an individual small Josephson tunnel junction. If the junction dynamics is confined to the lowest energy band of its band structure, it behaves as a variable capacitance with respect to the injected charge [21]. The magnitude of this capacitance depends on the bias point and can even be made *negative*. This is important for the application as a controllable coupling element, since it enables one to drive the coupling capacitance to zero. The physics of the capacitance modulation is the transformation of charge, when an injected charge causes a Cooper-pair transfer across the junction that changes the output charge. If, however, the two qubits are coupled directly through a small junction (the coupling scheme considered, e.g., in a somewhat different context in [22,23]) this mechanism would not produce a variable-capacitance coupling scheme, since variations of the junction capacitance are compensated by a redistribution of the transferred charge on the capacitances in series with the junction. Nevertheless, if the junction is included "perpendicular" to the coupling direction, as in Fig. 1, the required "variable electrostatic transformer" with a gate-voltage-controlled coupling capacitance is produced.

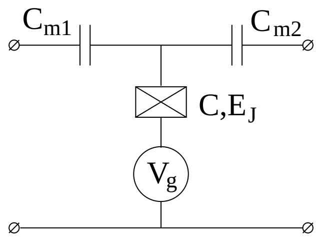

FIG. 1. Equivalent circuit of the variable electrostatic transformer for the controlled coupling of charge qubits. The Josephson coupling energy E_J of the small tunnel junction and the capacitances C, C_{m1}, C_{m2} of the structure determine the dispersion relation of the junction. Its lowest energy band $\epsilon_0(q)$ provides the required variable coupling capacitance controlled by the gate voltage V_g.

Indeed, if the voltage V is applied to the pair of input terminals (left or right, $i = 1, 2$) of the circuit shown in Fig. 1, the Hamiltonian of the system is that of an individual Josephson junction (or "Cooper-pair box"

[24,25]) with the charge $2e(q - q')$ injected into it, where $q = C_{mi}V/2e$ and $q' = C_{mi}V_g/2e$. When the junction is in its lowest energy band ϵ_0, the output voltage V_{out} at the opposite terminals varies such that

$$4e^2/C_0 \equiv 2e\partial V_{out}/\partial q = \partial^2\epsilon_0(q - q')/\partial q^2 \,. \tag{1}$$

The input-output interaction strength defined by this relation varies with the charge $2eq'$ injected by the gate voltage and goes through zero at a bias point dependent on the shape of the energy band $\epsilon_0(q)$.

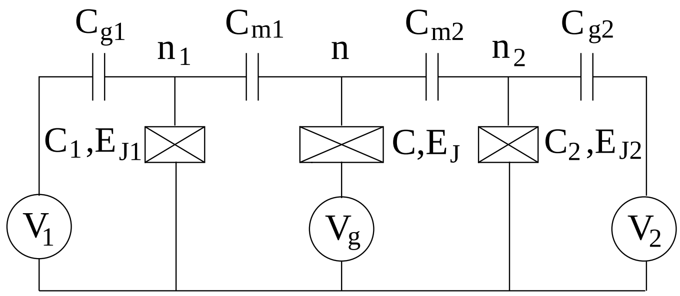

FIG. 2. Equivalent circuit of the two charge qubits coupled by a variable electrostatic transformer.

If the transformer shown in Fig. 1 is inserted between two charge qubits (Fig. 2), it provides a gate-voltage-controlled qubit coupling. The qubits 1,2 with Josephson coupling energies E_{Ji}, $i = 1, 2$, are biased by the gate voltages V_i through the capacitances C_{gi} and are coupled to the central island of the transformer by capacitances C_{mi}. Inverting the capacitance matrix of this system, we can write the Hamiltonian as

$$H = -\sum_{i=1,2} E_{Ji}\cos\varphi_i + E_{Ci}(n_i - q_i)^2 - E_J\cos\varphi + E_C(n - q(n_1, n_2))^2 \,, \tag{2}$$

where $E_C = 2e^2/(C_\Sigma - \sum_i C_{mi}^2/C_{\Sigma i})$, $E_{Ci} = 2e^2/C_{\Sigma i}$, n_i is the number of excess Cooper pairs on qubit i, and $q_i \equiv V_iC_{gi}/2e$. The qubits are coupled through the charge q induced on the transformer junction: $q = q_g - \sum_i(n_i - q_i)C_{mi}/C_{\Sigma i}$, where $q_g \equiv (V_g/2e)\sum_i C_{mi}(1 - C_{mi}/C_{\Sigma i})$. The Josephson phase differences across the respective junctions are denoted by φ_i and φ. Finally, $C_{\Sigma i} \equiv C_i + C_{gi} + C_{mi}$, and $C_\Sigma \equiv C + C_{m1} + C_{m2}$.

We assume that the degrees of freedom of the transformer junction are fast, and that the junction is confined to the lowest energy band of its band structure. Then we can replace the part of the Hamiltonian related to the transformer by the dispersion relation $\epsilon_0(q)$ of this lowest band,

$$-E_J\cos\varphi + E_C(n - q(n_1, n_2))^2 \to \epsilon_0(q(n_1, n_2)) \,. \tag{3}$$

This is the analogue of the Born-Oppenheimer approximation which in our case is valid if the characteristic energy gap between the bands of the transformer junction is much larger than the qubit energies. For our qubit coupling, this condition requires that $E_J \gg E_{Ji}$ for $E_J \ll E_C$, and $(E_CE_J)^{1/2} \gg E_{Ji}$ for $E_J \geq E_C$ (the qubits are assumed to be in the charging regime).

To determine the qubit coupling provided by the transformer we will use the known properties of the junction bandstructure [21,26]. Also, we use the fact that in the charge qubit regime ($E_{Ji} \ll E_{Ci}$, $q_i \simeq 1/2$), the dynamics of the charges n_i in (2) is reduced to two states, so that the charges can be expressed through the Pauli matrices, $n_i = (\sigma_{zi} + 1)/2$. Assuming that the structure in Fig. 2 is symmetric, i.e., $C_{m1}/C_{\Sigma 1} = C_{m2}/C_{\Sigma 2} \equiv c$, the transformer term $\epsilon_0(q(n_1, n_2))$ in the reduced Hamiltonian can be expressed as follows:

$$\epsilon_0(q(n_1, n_2)) = \nu\sigma_{z1}\sigma_{z2} + \delta(\sigma_{z1} + \sigma_{z2}) + \mu \,. \tag{4}$$

The coupling coefficient ν is given by

$$\nu = \frac{1}{4}[\epsilon_0(q_0 + c) + \epsilon_0(q_0 - c) - 2\epsilon_0(q_0)] \,, \tag{5}$$

where $q_0 = q_g + c\sum_i(q_i - 1/2)$. The term linear in σ_z shifts the qubit bias by $\delta = [\epsilon_0(q_0 + c) - \epsilon_0(q_0 - c)]/4$, and the constant term $\mu = [\epsilon_0(q_0 - c) + \epsilon_0(q_0 + c) + 2\epsilon_0(q_0)]$ affects the energy of all two-qubit states and is not relevant as long as we are discussing one pair of qubits.

Equation 5 is one of the central results of our work. We see that ν has the structure of a discretized second derivative of ϵ_0 similarly to Eq. (1). Since $\epsilon_0(q)$ is a periodic function, ν can be positive or negative. For instance, in the tight-binding limit $E_C \ll E_J$, when $\epsilon_0(q) = -\Delta\cos(2\pi q)$, the coupling is:

$$\nu = \frac{\Delta}{2}\cos(2\pi q_0)(1 - \cos(2\pi c)) \,, \tag{6}$$

and $\delta = (\Delta/2)\cos(2\pi q_0)\cos(2\pi c)$. Obviously, the coupling can be controlled by the gate voltage through the average induced charge q_0, and can change sign. In general, the coupling constant can be calculated numerically from Eq. (5). The results are plotted in Fig. 3 which shows that in accordance with Eq. (6), the q_0-dependence of ν is harmonic for $E_C \leq E_J$, and changes sign at $q_0 \approx \pm 1/4$. At larger E_C, $\nu(q_0)$ becomes non-harmonic, and for $E_C \gg E_J$ and small coupling coefficients c, approaches $c^2E_C/2$ for all q_0 except for the vicinity of the point $q_0 = \pm 1/2$, where ν/c^2E_C becomes large in absolute value and negative: $\nu/c^2E_C \simeq -E_C/2E_J$ for $c \ll E_J/E_C$, and $\nu/c^2E_C \simeq -1/2c$ for $E_J/E_C \ll c \ll 1$. The periodic nature of $\epsilon_0(q)$ which has a minimum at $q = 0$ and a maximum at $q = \pm 1/2$ implies that the maximum of the absolute value of ν is reached for $c = 1/2$ and is equal to half the bandwidth, e.g., $E_C/8$ for $E_C \gg E_J$, and Δ for $E_C \ll E_J$ – see Eq. (6).

The controlled electrostatic coupling of two qubits offers new possibilities to manipulate two-qubit states. The simplest one is the change of the structure of the *charging diagram* of the transformer-coupled qubits with the gate voltage V_g. The charging diagram shows the regions of stability of the charge states (n_1, n_2) as a function of the induced charges q_i, for small Josephson coupling $E_{Ji} \ll E_{Ci}$. The diagram is periodic with period 1 in both charges q_i, and has a honeycomb structure in the case of fixed electrostatic coupling. Figure 4 shows examples of the charging diagram. The diagram was obtained by minimizing the electrostatic qubit energy together with the coupling energy (3).

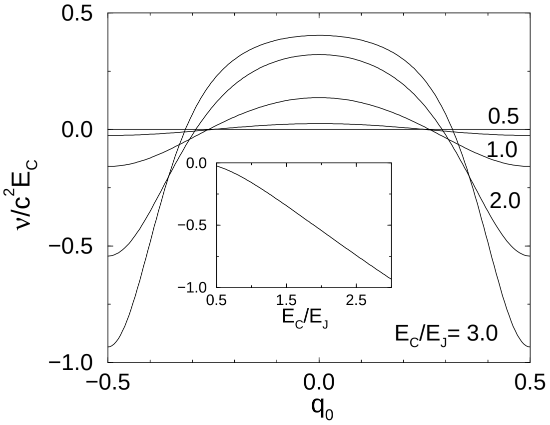

FIG. 3. Coupling energy ν of the two qubits in units of the charging energy E_C of the transformer junction as a function of the average induced charge q_0. The coupling strength is $c = 0.1$, and the ratio E_C/E_J is given for each curve. The normalization of ν by c^2 makes it independent of c for small c. Inset: $\nu(q_0)$ at $q_0 = \pm 0.5$, as a function of E_C/E_J for the same coupling strength c.

At $c = 0.3$, the main qualitative features of Fig. 4 can be understood neglecting the difference between q_0 and q_g, from the q_0-dependence of the qubit interaction, see Fig. 3. For $q_g = 0$, the coupling is positive and the charging diagram has the usual shape characteristic of a fixed electrostatic coupling. The cells of the honeycomb structure are tilted in the $q_1 = -q_2$ direction, since increasing the gate voltage of one qubit makes it easier to add a charge to the other qubit. In the absence of coupling, the degeneracy point of the four charge states $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ is extended in the $q_1 = q_2$ direction into the boundary between $(0,1)$, $(1,0)$, since positive coupling makes the charging energy of these two states lower than that of $(0,0)$, $(1,1)$. If the coupling is mostly negative, e.g., for $q_g = -0.5$, the cells of the honeycomb structure are tilted in the $q_1 = q_2$ direction. The degeneracy point of the four charge states is changed differently, since the states $(0,0)$, $(1,1)$ now have a lower charging energy than $(0,1)$, $(1,0)$. The degeneracy point is extended in the $q_1 = -q_2$ direction and creates a direct boundary between $(0,0)$ and $(1,1)$. Around $q_g = 0.3$, the qubit coupling is suppressed. Hence, four neighboring stability regions touch in one point, and their shape is approximately rectangular like for non-interacting qubits. In contrast to the case of fixed coupling, all the cells in Fig. 4 are curved, since changing q_1, q_2 also changes the effective charge q_0 of the transformer island, so that the effective qubit coupling constant ν and the renormalization δ of the qubit bias change with q_i. This effect is largest for values of q_g around which $\nu(q_0)$ varies strongly, like e.g., $q_g = -0.4$, see Figs. 3 and 4.

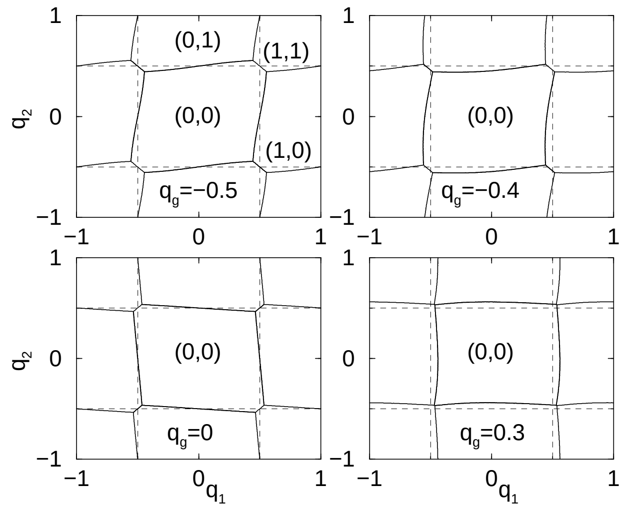

FIG. 4. Charging diagram of two charge qubits coupled through the variable electrostatic transformer for $c = 0.3$ and $E_C/E_J = 3$. The curves show the boundaries of the stability regions of the charge states (n_1, n_2) in the plane of the two induced charges $q_i \equiv V_i C_{gi}/2e$, $i = 1, 2$. The gate charge q_g is controlled by the transformer gate voltage V_g. The changing shape of the stability region reflects the V_g-controlled transition from positive to negative effective coupling capacitance of the two qubits. Dashed lines: uncoupled case.

The possibility to switch the coupling between two qubits on and off is important for the realization of practically all types of two-qubit gates. An advantage of our scheme is that the effective qubit coupling can be suppressed even in presence of a small direct positive geometric capacitance between the qubit islands. This "parasitic" capacitance only shifts the values of the gate voltages at which the total coupling vanishes. As an example, we show how our proposal can be used to directly realize a *phase gate*, a transformation that changes the sign of the $|11\rangle$ state and does not change the amplitudes of the other three two-qubit states. If the qubit tunneling amplitudes are tuned to zero, the Hamiltonian of the two-qubit system (Fig. 2) with coupling (4) is

$$H = \nu \sigma_{z1}\sigma_{z2} + \eta(\sigma_{z1} + \sigma_{z2}) , \tag{7}$$

where $\eta = \delta - (q_i - 1/2)E_{Ci}$. (For simplicity, we again discuss only the symmetric structure, which for the purpose of realizing the phase gate also implies identical qubit charges q_i. All results of our work can be directly extended to the asymmetric case.) By adjusting the three gate voltages V_1, V_2, and V_g such that $\nu(q_0 = q^*) = 0$ and $q_i = 1/2 + \delta(q^*)/E_{Ci}$ one can make the Hamiltonian (7) vanish. For $E_J \ll E_C$ such a stationary qubit state is realized when $q_i = 1/2$ and $q_0 = q_g = 1/4$. By applying gate-voltage pulses one can temporarily switch on both the interaction and the qubit bias in such a way that the qubit states accumulate dynamic phases [27]. To obtain a phase gate, i.e., to have the state $|11\rangle$ accumulate the phase π up to an irrelevant common phase of all four qubit states, the phase $\phi_\nu = \int dt\nu(t)$ due to the interaction energy and the phase $\phi_\eta = \int dt\eta(t)$ due to the qubit bias should be chosen as $\phi_\nu = \phi_\eta = \pi/4$.

These conditions fix only the time integral of the gate-voltage pulses. Their shape can be chosen to reduce the excitation amplitude of the upper energy band of the transformer junction that would invalidate our Born-Oppenheimer approximation (3) and effectively entangle the qubit states with the states of the transformer creating an additional decoherence mechanism. Transitions to the upper energy bands are suppressed if the pulses are sufficiently slow on the scale of the transformer energies. In this case, the amplitude $\alpha_k(t)$ of an excitation process to the kth band can be calculated by standard adiabatic perturbation theory as

$$\alpha_k = \int^t d\tau \frac{\langle k|\partial H/\partial\tau|0\rangle}{\epsilon_k(\tau)} e^{-i\int_\tau^t \epsilon_k(\tau')d\tau'}\,, \tag{8}$$

where ϵ_k is the energy of the kth band relative to ϵ_0. Since the total phases accumulated by the qubit states during the gate-voltage pulses are of the order of π, the adiabatic condition also means that the pulse amplitude should be small, i.e., $q_0 = q^* + x(t)$, $x \ll 1$. Therefore, the energies and matrix elements in Eq. (8) can be evaluated at $q_0 = q^*$, and Eq. (8) reduces to

$$|\alpha_k| = 2E_C|\langle k|(n-q)|0\rangle||x(\epsilon_k)|\,. \tag{9}$$

Here, $x(\epsilon_k) = \int d\tau x(\tau)e^{i\epsilon_k\tau}$ is the Fourier component of $x(t)$ which decreases exponentially in $\epsilon_k\tau_0$ for smooth pulses, where τ_0 is the characteristic pulse time. For $E_J \ll E_C$, the energy-band gap of the transformer is of the order of E_J, and this sets a lower limit for the pulse time through Eq. (9). For $E_J \geq E_C$, the gap between the energy bands grows and the adiabaticity condition is easier to satisfy (see the discussion after Eq. (3).

In conclusion, we have proposed a method to couple two Josephson charge qubits by an electrostatic transformer that produces an effective coupling that can be varied in magnitude and sign. We have given an explicit expression of this coupling as a discretized second derivative of the energy band of a small Josephson junction. Our proposal works even for asymmetric structures and in the presence of parasitic capacitances. It allows the implementation of a variety of two-qubit gates, and we have explicitly demonstrated how to build a phase gate.

C. B. would like to thank the University of Stony Brook for its hospitality during a one-month stay. The authors acknowledge discussions with K.K. Likharev which stimulated this work, and useful discussions of the results with J.E. Lukens and Yu.A. Pashkin. This work was supported in part by the NSF under grant # 0121428, by ARDA and DOD under the DURINT grant # F49620-01-1-0439 (D.V.A.), and by the Swiss NSF and the NCCR Nanoscience (C. B.).

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[27] The geometric phase that in general could also be created by adiabatic evolution is absent in this case since only one parameter, q_0, affects the transformer wavefunction.